Problem 1 (10+15+5=30 marks)
As usual, we denote by $A, A^\dagger$ the ladder operators of the harmonic oscillator, and by ω its circular frequency. In this problem we consider the dynamics governed by the Hamilton operator

$$H = \hbar\omega(A^\dagger - \alpha^*)(A - \alpha)\,,$$

where α is a complex constant.

(a) State the equations of motion obeyed by $A(t)$ and $A^\dagger(t)$ and solve them.

(b) Find the time transformation function $\langle a^*, t|a', t_0\rangle$.
Hint: What is its t derivative?

(c) If the system is in the $n = 0$ Fock state at time t_0, what is the probability of finding the system in the $n = 0$ Fock state at time t?

Problem 2 (10 marks)
Three dimensions: position vector operator $\vec{R}$, momentum vector operator $\vec{P}$, orbital angular momentum vector operator $\vec{L} = \vec{R} \times \vec{P}$. — A vector operator $\vec{F}$ is given by $\vec{F} = p\vec{R} + x\vec{P} + \vec{L}$, where p and x are numerical parameters. Express $\vec{F} \times \vec{F}$ as a linear combination of $\vec{R}$, $\vec{P}$, and $\vec{L}$.

Problem 3 (5+20=25 marks)
Orbital angular momentum vector operator $\vec{L}$ with cartesian components L_1, L_2, and L_3; as usual, $|l, m\rangle$ is a joint eigenket of $\vec{L}^{\,2}$ and L_3. — The system is in the $l = 2$ state described by the ket

$$|\ \rangle = |2, 2\rangle x + |2, 0\rangle y + |2, -2\rangle x$$

with real coefficients x and y.

(a) Which statement about x and y follows from the normalization $\langle\ |\ \rangle = 1$?

(b) Determine the spreads δL_1, δL_2, and δL_3.
Hint: $\delta L_2 = 0$ and $\delta L_1 = \delta L_3$ for $x = \sqrt{\frac{3}{8}}$, $y = \frac{1}{2}$.

Problem 4 (15 marks)
For hydrogenic atoms with Hamilton operator $H = \dfrac{\vec{P}^{\,2}}{2M} - \dfrac{Ze^2}{|\vec{R}|}$ show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{|\vec{R}|} = -\frac{1}{2}\left(\frac{\mathrm{d}\vec{R}}{\mathrm{d}t}\cdot\frac{\vec{R}}{|\vec{R}|^3} + \frac{\vec{R}}{|\vec{R}|^3}\cdot\frac{\mathrm{d}\vec{R}}{\mathrm{d}t}\right).$$

Problem 5 (20 marks)
A harmonic oscillator (Hamilton operator $H_0 = \hbar\omega A^\dagger A$) is perturbed by $H_1 = \lambda D^\dagger D$ with $D = 1 + (AA^\dagger)^{-1/2}A$, so that the nth eigenvalue $E_n(\lambda)$ of $H = H_0 + H_1$ is a function of the real strength parameter λ. Of course, we have the unperturbed energies $E_n(\lambda = 0) = n\hbar\omega$ with $n = 0, 1, 2, \ldots$. — Find $\left.\dfrac{\mathrm{d}}{\mathrm{d}\lambda}E_n(\lambda)\right|_{\lambda=0}$.