AY 2006/07, Sem 2, PC3130, Make-up test

Question. 1/5

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$\boxed{1}$ (a) $\frac{d}{dt} X = \frac{\partial H}{\partial P} = v$,

$$X(t) = X(t_0) + vT \quad \text{with } T = t - t_0,$$

$$\frac{d}{dt} P = -\frac{\partial H}{\partial X} = -\kappa^2 X$$

$$P(t) = P(t_0) - \kappa^2 T X(t_0) - \tfrac{1}{2}\kappa^2 v T^2.$$

(b) $i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \langle x, t | H | p, t_0 \rangle$

with $H = v P(t) + \frac{1}{2}\kappa^2 X(t)^2$

$$= v P(t_0) + \tfrac{1}{2}\kappa^2 X(t_0)^2$$

$$= v P(t_0) + \tfrac{1}{2}\kappa^2 \left(X(t) - vT\right)^2,$$

so that

$$i\hbar \frac{\partial}{\partial t} \log \langle x, t | p, t_0 \rangle$$

$$= v p + \tfrac{1}{2}\kappa^2 (x - vT)^2$$

$$= \frac{\partial}{\partial t}\left( vTp - \frac{\kappa^2}{6v}(x - vT)^3 \right)$$

implying

$$\langle x, t | p, t_0 \rangle = \underbrace{\frac{e^{ixp/\hbar}}{\sqrt{2\pi\hbar}}} e^{-ivTp/\hbar} e^{\frac{i\kappa^2}{6\hbar v}\left[(x - vT)^3 - x^3\right]}$$

where the prefactor ensures the correct value for $t = t_0$.

(c) Given the solutions of the equations of motion in (a), we have immediately

$$\langle X(t)\rangle = vT, \quad \langle P(t)\rangle = -\tfrac{1}{2}k^2 v T^2,$$

and

$$\langle X(t)^2\rangle = \langle X(t_0)^2\rangle + 2vT\langle X(t_0)\rangle + (vT)^2$$

$$= x_0^2 + (vT)^2,$$

so that

$$\delta X(t) = \sqrt{\langle X(t)^2\rangle - \langle X(t)\rangle^2}$$

$$= \sqrt{x_0^2} = |x_0| = x_0$$

assume $x_0 > 0$.

Further,

$$\langle P(t)^2\rangle = \langle P(t_0)^2\rangle + (k^2T)^2\langle X(t_0)^2\rangle$$

$$+ \left(\tfrac{1}{2}k^2 vT^2\right)^2$$

$$- k^2 T\left\langle\left(X(t_0)P(t_0) + P(t_0)X(t_0)\right)\right\rangle$$

$$- k^2 v T^2\langle P(t_0)\rangle$$

$$+ k^4 v T^3\langle X(t_0)\rangle$$

$$= p_0^2 + (k^2 T x_0)^2 + \left(\tfrac{1}{2}k^2 vT^2\right)^2$$

giving

$$\delta P(t) = \sqrt{p_0^2 + (k^2 T x_0)^2}\,.$$

$\boxed{2}$ Multiply by coherent state bra $\langle a'^{*}|$:

$$\langle a'^{*}|e^{zA^{\dagger}}|a\rangle = e^{za'^{*}}\langle a'^{*}|a\rangle$$

$$= e^{za'^{*}} e^{a'^{*}a} = e^{a'^{*}(a+z)} = \langle a'^{*}|a+z\rangle$$

and since the $\langle a'^{*}|$ bras are complete, we get

$$e^{zA^{\dagger}}|a\rangle = |a+z\rangle.$$

Similarly, $\langle a^{*}|e^{z^{*}A} = \langle a^{*}+z^{*}|.$

$\boxed{3}$ $\vec{R}\times\vec{F} = \vec{R}\times\frac{d}{dt}\vec{P} = \frac{d}{dt}(\vec{R}\times\vec{P}) - \frac{d\vec{R}}{dt}\times\vec{P}$

$$= \frac{d}{dt}(\vec{R}\times\vec{P}) - \frac{1}{M}\underbrace{\vec{P}\times\vec{P}}_{=0} = \frac{d}{dt}(\vec{R}\times\vec{P})$$

$$= \frac{1}{i\hbar}\left[\vec{R}\times\vec{P}, H\right]$$

and since the expectation value is taken for an eigenstate of $H$ itself, we have

$$\langle \vec{R}\times\vec{F}\rangle = 0.$$

$\boxed{4}$ (a) Since $L_1\cos\gamma + L_2\sin\gamma = \begin{pmatrix}L_1\\L_2\\L_3\end{pmatrix}\cdot\underbrace{\begin{pmatrix}\cos\gamma\\\sin\gamma\\0\end{pmatrix}}_{\text{unit vector}}$,

it is a cartesian component of $\vec{L}$ in the 12-plane (= xy plane) and, Question. 4/5

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therefore, the eigenvalues are the same as the eigenvalues of $L_3$, namely $0, \pm\hbar$.

(b) We have $L_1 = \frac{1}{2}(L_+ + L_-)$, $L_2 = \frac{1}{2i}(L_+ - L_-)$,

$$L_1 |1,1\rangle = |1,0\rangle \hbar/\sqrt{2},$$
$$L_2 |1,1\rangle = |1,0\rangle i\hbar/\sqrt{2},$$
$$L_1 |1,0\rangle = (|1,1\rangle + |1,-1\rangle)\hbar/\sqrt{2},$$
$$L_2 |1,0\rangle = (|1,1\rangle - |1,-1\rangle)(-i\hbar/\sqrt{2}),$$
$$L_1 |1,-1\rangle = |1,0\rangle \hbar/\sqrt{2},$$
$$L_2 |1,-1\rangle = |1,0\rangle(-i\hbar/\sqrt{2}),$$ so that

$$(L_1 \cos\gamma + L_2 \sin\gamma)|1,1\rangle = |1,0\rangle \frac{\hbar}{\sqrt{2}} e^{i\gamma},$$
$$(L_1 \cos\gamma + L_2 \sin\gamma)|1,0\rangle = |1,1\rangle \frac{\hbar}{\sqrt{2}} e^{-i\gamma} + |1,-1\rangle \frac{\hbar}{\sqrt{2}} e^{i\gamma},$$
$$(L_1 \cos\gamma + L_2 \sin\gamma)|1,-1\rangle = |1,0\rangle \frac{\hbar}{\sqrt{2}} e^{-i\gamma}.$$

It follows that the eigenket to eigenvalue 0 is
$$|ev=0\rangle = (|1,1\rangle e^{-i\gamma} - |1,-1\rangle e^{i\gamma})/\sqrt{2},$$

and the eigenkets to eigenvalues $\pm\hbar$ are
$$|ev=\pm 1\rangle = (|1,1\rangle e^{-i\gamma} \pm |1,0\rangle\sqrt{2} + |1,-1\rangle e^{i\gamma})/2.$$

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$\boxed{5}$ We have $H = H_0 + H_1$ with $H_0$ = Hamilton operator of hydrogenic atoms as in the notes, and

$$H_1 = V_1(|\vec{R}|) \quad \text{with}$$

$$V_1(r) = V_{\text{sphere}}(r) - V_{\text{point}}(r) = \begin{cases} \frac{Ze^2}{r} - \frac{Ze^2}{2b^3}(3b^2 - r^2) & \text{for } r < b, \\ 0 & \text{for } r > b. \end{cases}$$

Therefore, the 1st-order correction to the ground-state energy is

$$\langle \underset{n'\ell' m}{100} | H_1 | 100 \rangle = \int (d\vec{r})\, V_1(r) \underbrace{|\langle \vec{r} | 100 \rangle|^2}_{= \frac{1}{4\pi} R_{10}(r)^2}$$

$$= \int_0^b dr\, r^2 V_1(r)\, 4\left(\frac{Z}{a_0}\right)^3 \underbrace{e^{-2Zr/a_0}}_{\cong 1 \text{ since } b \ll a_0}$$

$$= 4\left(\frac{Z}{a_0}\right)^3 Ze^2 \int_0^b dr \left(r - \frac{3}{2}\frac{r^2}{b} + \frac{1}{2}\frac{r^4}{b^3}\right)$$

$$= 4\left(\frac{Z}{a_0}\right)^3 Ze^2 \left(\frac{1}{2}b^2 - \frac{1}{2}b^2 + \frac{1}{10}b^2\right)$$

$$= \frac{Z^2e^2}{2a_0}\,\frac{4}{5}\left(\frac{b}{a_0/Z}\right)^2,$$ and the corrected ground-state energy is

$$E_0 = -\frac{Z^2e^2}{2a_0}\left(1 - \frac{4}{5}\left(\frac{b}{a_0/Z}\right)^2\right).$$