Problem 1 (8+12+10=30 marks)
Consider the one-dimensional motion (position operator X, momentum operator P) that is governed by the Hamilton operator

$$H = vP + \frac{1}{2}\kappa^2 X^2$$

where v and κ are positive constants.

(a) State and solve the equations of motion obeyed by $X(t)$ and $P(t)$.

(b) Determine the time transformation function $\langle x, t|p, t_0\rangle$.

(c) Given the expectation values $\langle X\rangle = 0$, $\langle P\rangle = 0$, $\langle XP\rangle = \frac{1}{2}\mathrm{i}\hbar$, $\langle X^2\rangle = x_0^2$, and $\langle P^2\rangle = p_0^2$ at time t_0, find the spreads $\delta X(t)$ and $\delta P(t)$ at time t.

Problem 2 (10 marks)
Harmonic oscillator ladder operators $A, A^\dagger$; coherent state kets $|a\rangle$ and bras $\langle a^*|$. Show that $\mathrm{e}^{zA^\dagger}|a\rangle = |a+z\rangle$ for all complex numbers z. Which bra results from $\langle a^*|\mathrm{e}^{z^*A}$?

Problem 3 (10 marks)
A three-dimensional system (mass M, position vector operator $\vec{R}$, momentum vector operator $\vec{P}$) is in an eigenstate of the Hamilton operator $H = \vec{P}^{\,2}/(2M) + V(\vec{R}\,)$. Show that the mean torque, that is: the expectation value of the torque vector operator $\vec{R}\times\vec{F}$, is zero, where $\vec{F} = -\dfrac{\partial}{\partial\vec{R}}V(\vec{R}\,)$ is the force vector operator.

Problem 4 (5+20=25 marks)
Orbital angular momentum vector operator $\vec{L}$ with cartesian components L_1, L_2, and L_3; as usual, $|l, m\rangle$ is a joint eigenket of $\vec{L}^{\,2}$ and L_3.

(a) For $l = 1$ and a given value for the angle parameter γ, what are the eigenvalues of $L_1\cos\gamma + L_2\sin\gamma$?

(b) Find the respective eigenkets of $L_1\cos\gamma + L_2\sin\gamma$ as superpositions of the eigenkets of L_3 .

Problem 5 (25 marks)
One simplifying assumption in the derivation of the Bohr energies of hydrogen-like atoms is that we regard the atomic nucleus as a point charge. When we want to describe it rather as a homogeneously charged ball, we replace the potential energy

$$V_{\mathrm{point}}(r) = -\frac{Ze^2}{r} \qquad \text{by} \qquad V_{\mathrm{sphere}}(r) = \begin{cases} -\dfrac{Ze^2}{2b^3}(3b^2 - r^2) & \text{for } r < b\,, \\ -\dfrac{Ze^2}{r} & \text{for } r > b\,, \end{cases}$$

where $b > 0$ is the radius of the ball-shaped nucleus. Bear in mind that b is a tiny fraction of the Bohr radius a_0 (roughly $b/a_0 \cong 10^{-4}$), and determine the resulting 1st-order change in the atomic ground-state energy. — Hint: For hydrogenic wave functions see equations (5.2.27), (6.7.6), and (6.7.16) in the lecture notes.