1. Mean velocity (10 marks)

Three-dimensional motion: position vector operator $\vec{R}$, momentum vector operator $\vec{P}$. The system is in an eigenstate of the Hamilton operator $H(\vec{P},\vec{R})$. Show that the mean velocity, that is: the expectation value of the velocity vector operator $\vec{V} = \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{R}$, is zero.

2. Time-dependent spreads (25=12+8+5 marks)

At time $t = 0$, the initial position wave function of a one-dimensional harmonic oscillator (position operator X, momentum operator P, mass M, circular frequency ω) is given by

$$\psi(x) = \sqrt{\kappa}\,\mathrm{e}^{-\kappa\,|x|}$$

with $\kappa > 0$.

(a) Determine $\delta X(t)$ and $\delta P(t)$, the time-dependent spreads in position and momentum, respectively.

(b) Verify that Heisenberg's position-momentum uncertainty relation is obeyed at all times.

(c) For which value of κ is the uncertainty product $\delta X(t)\,\delta P(t)$ independent of time t?

3. Orbital angular momentum (20=5+10+5 marks)

The Hamilton operator of a spinning top is

$$H = \frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}L_2^2 + \frac{1}{2I_3}L_3^2\,,$$

where L_1, L_2, L_3 are the cartesian components of the angular momentum vector operator $\vec{L}$, and I_1, I_2, I_3 are the moments of inertia for the three major axes of rotation.

(a) State the equation of motion obeyed by $L_1(t)$.

(b) If the top is in a common eigenstate of $\vec{L}^{\,2}$ and L_3 with eigenvalues $2\hbar^2$ and $\hbar$, respectively, what is the expectation value $\langle H\rangle$ of H and what is its spread δH?

(c) If $I_2 = I_3$, what are the eigenvalues of H?

4. Hydrogen-like atoms (20=8+8+4 marks)

You have a tritium atom (^{3}H, nuclear charge $Z = 1$) in its ground state [principal quantum number $n = 1$, angular momentum quantum numbers $(l, m) = (0, 0)$]. Suddenly the triton nucleus undergoes a β decay whereby the emitted electron (and also the neutrino) escape so rapidly that we can regard the net effect as an instantaneous replacement of the triton by a ^{3}He nucleus (nuclear charge $Z = 2$). For the bound electron, this amounts to a sudden doubling of the nuclear charge.

(a) What is the probability that, after the decay, the resulting $^3\text{He}^+$ ion is found in its electronic ground state as well?

(b) What is the probability that you find the $^3\text{He}^+$ ion in its excited state with $n = 2$ and $l = 0$?

(c) What is the probability that you find the $^3\text{He}^+$ ion in one of its exited states with $n = 2$ and $l = 1$?

Hint: For hydrogenic wave functions see equations (5.2.27), (6.7.6), and (6.7.16) in the lecture notes.

5. Perturbation Theory (25=15+10 marks)

A harmonic oscillator (ladder operators $A, A^\dagger$; Hamilton operator $H_0 = \hbar\omega A^\dagger A$) is perturbed by $H_1 = \hbar\Omega[A^\dagger(AA^\dagger)^{-1/2} + (AA^\dagger)^{-1/2}A]$. We denote the nth eigenvalue of the total Hamilton operator $H = H_0 + H_1$ by $E_n = \hbar\omega\epsilon_n(\Omega/\omega)$ where, of course, the unperturbed energies $E_n^{(0)} = n\hbar\omega$ are recovered by $\epsilon_n(0) = n$ for $n = 0, 1, 2, \ldots$.

(a) For $n = 0, 1, 2, \ldots$, determine $\epsilon_n(\Omega/\omega)$ up to 2nd order in Ω/ω by Rayleigh–Schrödinger perturbation theory.

(b) Find the 2nd-order approximation to $\epsilon_0(\Omega/\omega)$ in Brillouin–Wigner perturbation theory (only $n = 0$ here).