AY 2007/08, Sem 1, PC3274, Test 1

Question 1/6

Write answers on this side of the paper only.

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Solution as written up by Teng Po-Wen, Tan Han Koon, and Teo Zhan Rui

1

Since we want to evaluate the smallest value (or extremum value) that is possible for the integral, we let

$$\left[\frac{d}{dx}y(x)\right]^2 + [y(x)]^3 = F, \tag{4}$$

which can also be expressed as

$$y'^2 + y^3 = F, \tag{5}$$

and hence applying the identity,

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}. \tag{6}$$

Substituting (5) into (6) :

$$y'^2 + y^3 - y'(2y') = \text{constant}. \tag{7}$$

However, as $x \to \infty$, y' and y are 0, therefore the constant is 0 too.

Simplifying,

$$y'^2 = y^3. \tag{8}$$

Therefore, we get a differential equation in which y' must be negative because y decreases as x increases :

$$\frac{dy}{dx} = -y^{\frac{3}{2}}. \tag{9}$$

Bringing the y term over,

$$\int -y^{-\frac{3}{2}}dy = \int dx\ . \tag{10}$$

Solving (10) gives,

$$2y^{-\frac{1}{2}} = x + c, \tag{11}$$

where c is a constant to be determined by boundary conditions.

When x=0, y=1, substituting this into (11) : c=2.

Therefore, squaring (11) and using the fact that c=2, we get:

Question 2/6

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$$y = \frac{4}{(x+2)^2}. \tag{12}$$

Substituting (12) into the integral below,

$$\int_0^\infty dx \left(\left[\frac{d}{dx} y(x)\right]^2 + [y(x)]^3 \right),$$

we get

$$\int_0^\infty dx \left(\left[-\frac{8}{(x+2)^3}\right]^2 + \left[\frac{4}{(x+2)^2}\right]^3 \right), \tag{13}$$

which can be evaluated as

$$\int_0^\infty dx \left(\frac{64}{(x+2)^6} + \frac{64}{(x+2)^6} \right). \tag{14}$$

Evaluating (14), it becomes

$$-\frac{128}{5(x+2)^5}\Big|_0^\infty = \frac{4}{5}. \tag{15}$$

Hence the smallest value for the integral is $\frac{4}{5}$.

2 In Cartesian coordinates, $L(z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$.

Using the given parameterization,

$$\begin{aligned} z &= \sqrt{x^2 + y^2 + a^2} \\ &= \sqrt{a^2 \sinh^2 \varsigma \cos^2 \varphi + a^2 \sinh^2 \varsigma \sin^2 \varphi + a^2} \\ &= a\sqrt{\sinh^2 \varsigma + 1} \\ &= a \cosh \varsigma. \end{aligned}$$

Then,

$$\dot{x} = a \cosh \varsigma \cos\varphi\, \dot{\varsigma} - a \sinh \varsigma \sin\varphi\, \dot{\varphi}, \tag{16}$$

$$\dot{y} = a \cosh \varsigma \sin\varphi\, \dot{\varsigma} + a \sinh \varsigma \cos\varphi\, \dot{\varphi}, \tag{17}$$

$$\dot{z} = a \sinh \varsigma\, \dot{\varsigma}, \tag{18}$$

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=a^2\cosh^2\varsigma\cos^2\varphi\,\dot{\varsigma}^2+a^2\sinh^2\varsigma\,\sin^2\varphi\,\dot{\varphi}^2 \\ +a^2\cosh^2\varsigma\sin^2\varphi\,\dot{\varsigma}^2+a^2\sinh^2\varsigma\cos^2\varphi\,\dot{\varphi}^2 \\ +a^2\sinh^2\varsigma\,\dot{\varsigma}^2 \\ =a^2(\cosh^2\varsigma+\sinh^2\varsigma)\dot{\varsigma}^2+a^2\sinh^2\varsigma\,\dot{\varphi}^2\,. \tag{19}$$

Note that the terms containing ($\dot{\varsigma}\dot{\varphi}$) in (19) cancel out.

So in the new coordinates,

$$\boxed{L(\varsigma,\dot{\varsigma},\dot{\varphi})=\frac{1}{2}ma^2[(\cosh^2\varsigma+\sinh^2\varsigma)\dot{\varsigma}^2+\sinh^2\varsigma\,\dot{\varphi}^2]-mga\cosh\varsigma.} \tag{20}$$

To find Hamilton function, we need p_ς and p_φ,

$$p_\varsigma=\frac{\partial L}{\partial\dot{\varsigma}}=ma^2(\cosh^2\varsigma+\sinh^2\varsigma)\dot{\varsigma}, \tag{21}$$

$$\Rightarrow \quad \dot{\varsigma}=\frac{p_\varsigma}{ma^2(\cosh^2\varsigma+\sinh^2\varsigma)}, \tag{22}$$

$$p_\varphi=\frac{\partial L}{\partial\dot{\varphi}}=ma^2\sinh^2\varsigma\,\dot{\varphi}, \tag{23}$$

$$\Rightarrow \quad \dot{\varphi}=\frac{p_\varphi}{ma^2\sinh^2\varsigma}. \tag{24}$$

Substituting $\dot{\varsigma}$ and $\dot{\varphi}$ into

$$H=p_\varsigma\dot{\varsigma}+p_\varphi\dot{\varphi}-L\,, \tag{25}$$

Finally,

$$\boxed{H(\varsigma,\,p_\varsigma,p_\varphi)=\frac{p_\varsigma^2}{2ma^2(\cosh^2\varsigma+\sinh^2\varsigma)}+\frac{p_\varphi^2}{2ma^2\sinh^2\varsigma}+mga\cosh\varsigma.} \tag{26}$$

Equations of motion:

$$\frac{dp_\varphi}{dt}=-\frac{\partial H}{\partial\varphi}=0\,, \tag{27}$$

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$$\frac{dp_\varsigma}{dt}=-\frac{\partial H}{\partial \varsigma}$$
$$=\frac{p_\varsigma^2}{2ma^2}\frac{4\sinh\varsigma\cosh\varsigma}{(\cosh^2\varsigma+\sinh^2\varsigma)^2}+\frac{p_\varphi^2(2\operatorname{cosech}^2\varsigma\coth\varsigma)}{2ma^2}-mga\sinh\varsigma$$
$$=\frac{p_\varsigma^2}{ma^2}\frac{2\sinh\varsigma\cosh\varsigma}{(\cosh^2\varsigma+\sinh^2\varsigma)^2}+\frac{p_\varphi^2\operatorname{cosech}^2\varsigma\coth\varsigma}{ma^2}-mga\sinh\varsigma, \tag{28}$$

$$\frac{d\varphi}{dt}=-\frac{\partial H}{\partial p_\varphi}=\frac{p_\varphi}{ma^2\sinh^2\varsigma}, \tag{29}$$

$$\frac{d\varsigma}{dt}=-\frac{\partial H}{\partial p_\varsigma}=\frac{p_\varsigma}{ma^2(\cosh^2\varsigma+\sinh^2\varsigma)}. \tag{30}$$

3 We know that $\frac{d}{dt}\rho = 0$, so by the general form of Hamilton's equation of motion,

$$\frac{d}{dt}\rho = \frac{d}{dt}\rho(x(t),p(t),t) = \frac{\partial\rho}{\partial x}\frac{dx(t)}{dt}+\frac{\partial\rho}{\partial p}\frac{dp(t)}{dt}+\frac{\partial\rho}{\partial t} \tag{31}$$
$$=\{\rho,H\}+\frac{\partial\rho}{\partial t}=0.$$

Hence, by Liouville's equation

$$\frac{\partial\rho}{\partial t}=-\{\rho,H\}=\frac{\partial\rho}{\partial p}\frac{\partial H}{\partial x}-\frac{\partial\rho}{\partial x}\frac{\partial H}{\partial p}. \tag{32}$$

Taking time derivative of the function in question,

$$\frac{d}{dt}\left(\int dxdp\,\rho(x,p,t)\right)=\int dxdp\frac{\partial}{\partial t}\rho(x,p,t), \tag{33}$$

Substituting equation (32) into equation (33),

$$\frac{d}{dt}\left(\int dxdp\,\rho(x,p,t)\right)=\int dxdp\left(\frac{\partial\rho}{\partial p}\frac{\partial H}{\partial x}-\frac{\partial\rho}{\partial x}\frac{\partial H}{\partial p}\right)$$

$$=\int dxdp\,\frac{\partial\rho}{\partial p}\frac{\partial H}{\partial x}-\int dxdp\,\frac{\partial\rho}{\partial x}\frac{\partial H}{\partial p}$$

$$= \int dx \left([\rho \frac{\partial H}{\partial x}]_{p=-\infty}^{p=\infty} - \int dp\, \rho \frac{\partial^2 H}{\partial p \partial x}\right) - \int dp \left([\rho \frac{\partial H}{\partial p}]_{x=-\infty}^{x=\infty} \int dx\, \rho \frac{\partial^2 H}{\partial x \partial p}\right)$$

$$= -\int dxdp\, \rho \frac{\partial^2 H}{\partial p \partial x} + \int dpdx\, \rho \frac{\partial^2 H}{\partial x \partial p}$$

$$= \int dxdp\, \rho \left(-\frac{\partial^2 H}{\partial p \partial x} + \frac{\partial^2 H}{\partial x \partial p}\right) = 0.$$

Hence, it can be seen that $\int dx\, dp\, \rho(x, p, t)$ is independent of time, t, and that integration covers the whole phase space.

4

For iso-z lines, we have the following differential equations,

$$\frac{dx}{x+z} = \frac{dy}{2} \tag{34}$$

Next, we evaluate in terms of *x*,*y*,*z*.

$$2\ln|x+z| - y = \text{constant}. \tag{35}$$

The fact that we can integrate while treating *z* as a constant is because we are using the iso-z lines. Therefore simplifying (35) :

$$(x+z)e^{-\frac{1}{2}y} = \text{constant}. \tag{36}$$

We now let *z*(*x*,*y*) = *u*($(x+z)e^{-\frac{1}{2}y}$), where *u* is an arbitary function.

Using the fact that *z*(*x*,0)=*x*, we get,

$$z(x,0) = x = u(2x). \tag{37}$$

Upon letting $2x = a$, (37) will now become

$$\frac{1}{2}a = u(a). \tag{38}$$

Now by letting $a = (x+z)e^{-\frac{1}{2}y}$, we can get the solution for *z*(*x*,*y*),

$$u\left((x+z)e^{-\frac{1}{2}y}\right) = \frac{1}{2}(x+z)e^{-\frac{1}{2}y} = z(x,y), \tag{39}$$

$$z = \frac{x}{2e^{\frac{1}{2}y} - 1}. \tag{40}$$

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Now, to verify whether this solution is correct or not, we substitute (40) into

$$\left[(x+z)\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right]z=0\,.$$

First, we find

$$\frac{\partial z}{\partial x}=\frac{1}{2e^{\frac{1}{2}y}-1}=\frac{z}{x}, \tag{41}$$

$$\frac{\partial z}{\partial y}=-\frac{xe^{\frac{1}{2}y}}{(2e^{\frac{1}{2}y}-1)^2}=-\frac{z^2e^{\frac{1}{2}y}}{x}. \tag{42}$$

We now try to evaluate the 2 terms in the qIPDE replacing *x* using (40),

$$(x+z)\frac{\partial z}{\partial x}=\left(z+\frac{z^2}{x}\right)=\left(z+\frac{z}{2e^{\frac{1}{2}y}-1}\right), \tag{43}$$

$$2\frac{\partial z}{\partial y}=-\frac{2z^2e^{\frac{1}{2}y}}{x}=-\frac{2ze^{\frac{1}{2}y}}{2e^{\frac{1}{2}y}-1}. \tag{44}$$

Now, by summing (43) and (44), we should get 0, if not z will be incorrect,

(43 + 44): (45)

$$z+\frac{z}{2e^{\frac{1}{2}y}-1}-\frac{2ze^{\frac{1}{2}y}}{2e^{\frac{1}{2}y}-1}=\frac{2ze^{\frac{1}{2}y}-z+z-2ze^{\frac{1}{2}y}}{2e^{\frac{1}{2}y}-1}=0\,.$$

Hence, we verified that

$$\boxed{z=\frac{x}{2e^{\frac{1}{2}y}-1}.}$$