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1

Since we want to evaluate the smallest value (or extremum value) that is possible for the integral, we let

$$\left[ \frac{d}{dx} y(x) \right]^2 + [y(x)]^3 = F, \quad (4)$$

which can also be expressed as

$$y'^2 + y^3 = F, \quad (5)$$

and hence applying the identity,

$$F - y' \frac{\partial F}{\partial y'} = \text{constant}. \quad (6)$$

Substituting (5) into (6):

$$y'^2 + y^3 - y'(2y') = \text{constant}. \quad (7)$$

However, as  $x \rightarrow \infty$ ,  $y'$  and  $y$  are 0, therefore the constant is 0 too.

Simplifying,

$$y'^2 = y^3. \quad (8)$$

Therefore, we get a differential equation in which  $y'$  must be negative because  $y$  decreases as  $x$  increases:

$$\frac{dy}{dx} = -y^{\frac{3}{2}}. \quad (9)$$

Bringing the  $y$  term over,

$$\int -y^{-\frac{3}{2}} dy = \int dx. \quad (10)$$

Solving (10) gives,

$$2y^{-\frac{1}{2}} = x + c, \quad (11)$$

where  $c$  is a constant to be determined by boundary conditions.

When  $x=0$ ,  $y=1$ , substituting this into (11):  $c=2$ .

Therefore, squaring (11) and using the fact that  $c=2$ , we get:

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$$y = \frac{4}{(x+2)^2}. \quad (12)$$

Substituting (12) into the integral below,

$$\int_0^{\infty} dx \left( \left[ \frac{d}{dx} y(x) \right]^2 + [y(x)]^3 \right),$$

we get

$$\int_0^{\infty} dx \left( \left[ -\frac{8}{(x+2)^3} \right]^2 + \left[ \frac{4}{(x+2)^2} \right]^3 \right), \quad (13)$$

which can be evaluated as

$$\int_0^{\infty} dx \left( \frac{64}{(x+2)^6} + \frac{64}{(x+2)^6} \right). \quad (14)$$

Evaluating (14), it becomes

$$-\frac{128}{5(x+2)^5} \Big|_0^{\infty} = \frac{4}{5}. \quad (15)$$

Hence the smallest value for the integral is  $\frac{4}{5}$ .

[2]

In Cartesian coordinates,  $L(z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$ .

Using the given parameterization,

$$\begin{aligned} z &= \sqrt{x^2 + y^2 + a^2} \\ &= \sqrt{a^2 \sinh^2 \zeta \cos^2 \varphi + a^2 \sinh^2 \zeta \sin^2 \varphi + a^2} \\ &= a \sqrt{\sinh^2 \zeta + 1} \\ &= a \cosh \zeta. \end{aligned}$$

Then,

$$\dot{x} = a \cosh \zeta \cos \varphi \dot{\zeta} - a \sinh \zeta \sin \varphi \dot{\varphi}, \quad (16)$$

$$\dot{y} = a \cosh \zeta \sin \varphi \dot{\zeta} + a \sinh \zeta \cos \varphi \dot{\varphi}, \quad (17)$$

$$\dot{z} = a \sinh \zeta \dot{\zeta}, \quad (18)$$

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$$\begin{aligned}
 \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= a^2 \cosh^2 \zeta \cos^2 \varphi \dot{\zeta}^2 + a^2 \sinh^2 \zeta \sin^2 \varphi \dot{\varphi}^2 \\
 &\quad + a^2 \cosh^2 \zeta \sin^2 \varphi \dot{\zeta}^2 + a^2 \sinh^2 \zeta \cos^2 \varphi \dot{\varphi}^2 \\
 &\quad + a^2 \sinh^2 \zeta \dot{\zeta}^2 \\
 &= a^2 (\cosh^2 \zeta + \sinh^2 \zeta) \dot{\zeta}^2 + a^2 \sinh^2 \zeta \dot{\varphi}^2.
 \end{aligned} \tag{19}$$

Note that the terms containing  $(\dot{\zeta}\dot{\varphi})$  in (19) cancel out.

So in the new coordinates,

$$L(\zeta, \dot{\zeta}, \dot{\varphi}) = \frac{1}{2} m a^2 [(\cosh^2 \zeta + \sinh^2 \zeta) \dot{\zeta}^2 + \sinh^2 \zeta \dot{\varphi}^2] - m g a \cosh \zeta. \tag{20}$$

To find Hamilton function, we need  $p_\zeta$  and  $p_\varphi$ ,

$$p_\zeta = \frac{\partial L}{\partial \dot{\zeta}} = m a^2 (\cosh^2 \zeta + \sinh^2 \zeta) \dot{\zeta}, \tag{21}$$



$$\dot{\zeta} = \frac{p_\zeta}{m a^2 (\cosh^2 \zeta + \sinh^2 \zeta)}, \tag{22}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m a^2 \sinh^2 \zeta \dot{\varphi}, \tag{23}$$



$$\dot{\varphi} = \frac{p_\varphi}{m a^2 \sinh^2 \zeta}. \tag{24}$$

Substituting  $\dot{\zeta}$  and  $\dot{\varphi}$  into

$$H = p_\zeta \dot{\zeta} + p_\varphi \dot{\varphi} - L, \tag{25}$$

Finally,

$$H(\zeta, p_\zeta, p_\varphi) = \frac{p_\zeta^2}{2 m a^2 (\cosh^2 \zeta + \sinh^2 \zeta)} + \frac{p_\varphi^2}{2 m a^2 \sinh^2 \zeta} + m g a \cosh \zeta. \tag{26}$$

Equations of motion:

$$\frac{d p_\varphi}{d t} = - \frac{\partial H}{\partial \varphi} = 0, \tag{27}$$

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$$\begin{aligned}
 \frac{dp_\zeta}{dt} &= -\frac{\partial H}{\partial \zeta} \\
 &= \frac{p_\zeta^2}{2ma^2} \frac{4\sinh \zeta \cosh \zeta}{(\cosh^2 \zeta + \sinh^2 \zeta)^2} + \frac{p_\phi^2 (2\operatorname{cosech}^2 \zeta \coth \zeta)}{2ma^2} - mga \sinh \zeta \\
 &= \frac{p_\zeta^2}{ma^2} \frac{2\sinh \zeta \cosh \zeta}{(\cosh^2 \zeta + \sinh^2 \zeta)^2} + \frac{p_\phi^2 \operatorname{cosech}^2 \zeta \coth \zeta}{ma^2} - mga \sinh \zeta, \quad (28)
 \end{aligned}$$

$$\frac{d\phi}{dt} = -\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ma^2 \sinh^2 \zeta} \quad (29)$$

$$\frac{d\zeta}{dt} = -\frac{\partial H}{\partial p_\zeta} = \frac{p_\zeta}{ma^2 (\cosh^2 \zeta + \sinh^2 \zeta)}. \quad (30)$$

3 We know that  $\frac{d}{dt}\rho = 0$ , so by the general form of Hamilton's equation of motion,

$$\begin{aligned}
 \frac{d}{dt}\rho &= \frac{d}{dt}\rho(x(t), p(t), t) = \frac{\partial \rho}{\partial x} \frac{dx(t)}{dt} + \frac{\partial \rho}{\partial p} \frac{dp(t)}{dt} + \frac{\partial \rho}{\partial t} \\
 &= \{\rho, H\} + \frac{\partial \rho}{\partial t} = 0. \quad (31)
 \end{aligned}$$

Hence, by Liouville's equation

$$\frac{\partial \rho}{\partial t} = -\{\rho, H\} = \frac{\partial \rho}{\partial p} \frac{\partial H}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial H}{\partial p}. \quad (32)$$

Taking time derivative of the function in question,

$$\frac{d}{dt} \left( \int dx dp \rho(x, p, t) \right) = \int dx dp \frac{\partial}{\partial t} \rho(x, p, t), \quad (33)$$

Substituting equation (32) into equation (33),

$$\begin{aligned}
 \frac{d}{dt} \left( \int dx dp \rho(x, p, t) \right) &= \int dx dp \left( \frac{\partial \rho}{\partial p} \frac{\partial H}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial H}{\partial p} \right) \\
 &= \int dx dp \frac{\partial \rho}{\partial p} \frac{\partial H}{\partial x} - \int dx dp \frac{\partial \rho}{\partial x} \frac{\partial H}{\partial p}
 \end{aligned}$$

