AY 2007/08, Sem 1, PC3274, Test 2

Question ...1/9....

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1

The two 2 × 2 matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

are elements of a matrix group with just a few group elements. By considering $S^{-1}, R^{-1}, S^2, SR, RS, R^2, \ldots$, find the other group elements. Is the group abelian? If it isn't, identify the abelian subgroups.

Solution:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$R = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

$$S^{-1} = -1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$S^{-1} = S,$$

$$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I,$$

where I is the identity matrix,

$$R^{-1} = 2 \cdot \frac{1}{(-1)(-1)-(-\sqrt{3})(\sqrt{3})}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

$$= 2 \cdot \frac{1}{4}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

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$$R^2 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1-3 & \sqrt{3}+\sqrt{3} \\ -\sqrt{3}-\sqrt{3} & -3+1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

$$R^2 = R^{-1},$$

notice that

$$RR^{-1} = I$$

$$= R^3,$$

Implying

$$(R^{-1})^2 = R.$$

$$SR = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix},$$

$$RS = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

$$R^{-1}S = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix},$$

$$R^{-1}S = SR,$$

notice that, by multiplying the above with S on both side, we get

$$SR^{-1}SS = SSRS$$

$$SR^{-1}I = IRS$$

$$SR^{-1} = RS.$$

	R	R^2	S	RS	SR	I
R	R^2	I	RS	SR	S	R
R^2	I	R	SR	S	RS	R^2
S	SR	RS	I	R^2	R	S
RS	S	SR	R	I	R^2	RS
SR	RS	S	R^2	R	I	SR
I	R	R^2	S	RS	SR	I

From the table, we can deduce that R, R^{-1}, S, RS, SR, I are the group elements. The group is not abelian.

Subgroups which are abelian are $\{R, R^2, I\}$, $\{S, I\}$, $\{RS, I\}$, $\{SR, I\}$, and $\{I\}$.

2

The set G consists of all complex 2×2 matrices $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ whose matrix elements are restricted by the relations

$$|M_{11}|^2 = |M_{22}|^2 = 1 + |M_{21}|^2 = 1 + |M_{12}|^2, \qquad M_{21}^*M_{11} = M_{12}M_{22}^*,$$

Demonstrate that $M_{11}^*M_{12} = M_{22}M_{21}^*$, and then show that G is a group with matrix multiplication as the group composition law.

Solution:

Given

$$M_{21}^*M_{11} = M_{12}M_{22}^*,$$

$$M_{21}M_{11}^* = M_{12}^*M_{22},$$

so,

$$M_{21}^*M_{12}M_{21}M_{11}^* = M_{21}^*M_{12}M_{12}^*M_{22}^*$$

$$|M_{21}|^2 M_{12}M_{11}^* = |M_{12}|^2 M_{22}M_{21}^*$$

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$$\therefore M_{12}M_{11}^* = M_{22}M_{21}^*.$$

(i) To prove <u>closure</u>, there are two methods, as follows:

Method 1

$$|M_{11}|^2 = 1 + |M_{21}|^2$$

$$M_{11}^* M_{12} = \frac{M_{12}}{M_{11}} + \frac{M_{12}}{M_{11}}|M_{21}|^2,$$

$$|M_{22}|^2 = 1 + |M_{21}|^2$$

$$M_{22}M_{21}^* = \frac{M_{21}^*}{M_{22}^*} + \frac{M_{21}^*}{M_{22}^*}|M_{21}|^2,$$

$$A = \begin{pmatrix} M_{11}^* & -M_{21}^* \\ -M_{12}^* & M_{22}^* \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$AM = \begin{pmatrix} M_{11}^* & -M_{21}^* \\ -M_{12}^* & M_{22}^* \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11}^*M_{11} - M_{21}^*M_{21} & M_{11}^*M_{12} - M_{21}^*M_{22} \\ -M_{12}^*M_{11} + M_{22}^*M_{21} & -M_{12}^*M_{12} + M_{22}^*M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I.$$

Hence, A is the unique inverse of M.

$$\therefore M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $N \in G$,

$$(MN)^{-1} = N^{-1}M^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N^+ M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (MN)^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, elements of Group G are closed under composition.

Method 2

Let

$$M' = \begin{pmatrix} M'_{11} & M'_{12} \\ M'_{21} & M'_{22} \end{pmatrix} \in G,$$

and

$$N = MM' = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

$$N_{11} = M_{11}M'_{11} + M_{12}M'_{21},$$

$$N_{12} = M_{11}M'_{12} + M_{12}M'_{22},$$

$$N_{21} = M_{21}M'_{11} + M_{22}M'_{21},$$

$$N_{22} = M_{21}M'_{12} + M_{22}M'_{22}.$$

$$|N_{11}|^2 = |M_{11}|^2|M'_{11}|^2 + |M_{12}|^2|M'_{21}|^2 + M_{11}M'_{11}M^*_{12}M'^*_{21} + M_{12}M'_{21}M^*_{11}M'^*_{11},$$

$$|N_{22}|^2 = |M_{21}|^2|M'_{12}|^2 + |M_{22}|^2|M'_{22}|^2 + M_{21}M'_{12}M^*_{22}M'^*_{22} + M_{22}M'_{22}M^*_{21}M'^*_{12},$$

$$|N_{12}|^2 = |M_{11}|^2|M'_{12}|^2 + |M_{12}|^2|M'_{22}|^2 + M_{11}M'_{12}M^*_{12}M'^*_{22} + M_{12}M'_{22}M^*_{11}M'^*_{12},$$

$$|N_{21}|^2 = |M_{21}|^2|M'_{11}|^2 + |M_{22}|^2|M'_{21}|^2 + M_{21}M'_{11}M^*_{22}M'^*_{21} + M_{22}M'_{21}M^*_{21}M'^*_{11},$$

$$\begin{aligned}|N_{11}|^2 - |N_{22}|^2 &= (|M_{11}|^2|M'_{11}|^2 - |M_{22}|^2|M'_{22}|^2) + (|M_{12}|^2|M'_{21}|^2 - |M_{21}|^2|M'_{12}|^2) \\ &\quad + (M_{11}M'_{11}M^*_{12}M'^*_{21} - M_{21}M'_{12}M^*_{22}M'^*_{22}) \\ &\quad + (M_{12}M'_{21}M^*_{11}M'^*_{11} - M_{22}M'_{22}M^*_{21}M'^*_{12}) \\ &= 0.\end{aligned}$$

Hence $|N_{11}|^2 = |N_{22}|^2$.

$$|N_{12}|^2 - |N_{21}|^2 = (|M_{11}|^2|M'_{12}|^2 - |M_{21}|^2|M'_{11}|^2) + (|M_{12}|^2|M'_{22}|^2 - |M_{22}|^2|M'_{21}|^2)$$
$$+(M_{11}M'_{12}M^*_{12}M'^*_{22} - M_{21}M'_{11}M^*_{22}M'^*_{21})$$
$$+(M_{12}M'_{22}M^*_{11}M'^*_{12} - M_{22}M'_{21}M^*_{21}M'^*_{11})$$
$$= 0.$$

Hence $|N_{12}|^2 = |N_{21}|^2$.

$$|N_{11}|^2 - |N_{12}|^2 = |M_{11}|^2(|M'_{11}|^2 - |M'_{12}|^2) + |M_{12}|^2(|M'_{21}|^2 - |M'_{22}|^2)$$
$$+M_{11}M^*_{12}(M'_{11}M'^*_{21} - M'_{12}M'^*_{22}) + M_{12}M^*_{11}(M'_{21}M'^*_{11} - M'_{22}M'^*_{12})$$
$$= |M_{11}|^2 - |M_{12}|^2$$
$$= 1.$$

Hence $|N_{11}|^2 = 1 + |N_{12}|^2$.

$$N^*_{21}N_{11} - N_{12}N^*_{22} = (M^*_{21}M'^*_{11} + M^*_{22}M'^*_{21})(M_{11}M'_{11} + M_{12}M'_{21})$$
$$-(M_{11}M'_{12} + M_{12}M'_{22})(M^*_{21}M'^*_{12} + M^*_{22}M'^*_{22})$$
$$= M_{11}M^*_{21}(|M'_{11}|^2 - |M'_{12}|^2) + M_{11}M^*_{22}(M'_{11}M'^*_{21} + M'_{12}M'^*_{22})$$
$$+M^*_{21}M_{12}(M'^*_{11}M'_{21} + M'^*_{12}M'_{22}) + M_{22}M^*_{12}(|M'_{21}|^2 - |M'_{22}|^2)$$
$$= M_{11}M^*_{21} - M_{22}M^*_{12}$$
$$= 0.$$

Hence $N^*_{21}N_{11} = N_{12}N^*_{22}$.

∴ The closure condition is satisfied.

(ii) <u>Neutral element</u>

The neutral element of the group is the identity matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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It can be easily shown that I satisfies the properties of an element of G.

$\therefore$ The neutral element is an element of G.

(iii) Associativity law

Since the group composition law is matrix multiplication, it is associative.

Hence, set G is a group.

3

Function $f(t)$ obeys the differential equation

$$\left(\frac{d^2}{dt^2} - 3\frac{d}{dt} + 2\right) f(t) = 2$$

and has the $t = 0$ values $f(0) = 1$ and $\frac{d}{dt}f(0) = 2$. First find the Laplace transform $F(s)$ of $f(t)$, and then $f(t)$ itself.

Solution:

$$\left(\frac{d^2}{dt^2} - 3\frac{d}{dt} + 2\right) f(t) = 2,$$

$$\frac{d^2}{dt^2} f(t) - 3\frac{d}{dt} f(t) + 2f(t) = 2,$$

$$s^2F(s) - sf(0) - \frac{d}{dt}f(0) - 3\big(sF(s) - f(0)\big) + 2F(s) = \frac{2}{s},$$

$$F(s)(s^2 - 3s + 2) + (-s + 3)f(0) - \frac{d}{dt}f(0) = \frac{2}{s},$$

but

$$\frac{d}{dt}f(0) = 2,$$

$$f(0) = 1,$$

$$F(s)(s^2 - 3s + 2) - s + 1 = \frac{2}{s},$$

$$F(s)(s-2)(s-1) = \frac{2}{s} + s - 1,$$

$$F(s) = \frac{1}{(s-1)(s-2)}\left(\frac{2}{s} + (s-1)\right)$$

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$$= \frac{2}{s(s-1)(s-2)} + \frac{1}{s-2}$$

$$= \frac{2}{s-2} - \frac{2}{s-1} + \frac{1}{s}$$

$$= 2H(s) - 2K(s) + G(s),$$

$$G(s) = \frac{1}{s} \to g(t) = 1,$$

$$H(s) = \frac{1}{s-2} \to h(t) = e^{2t},$$

$$K(s) = \frac{1}{s-1} \to k(t) = e^{t},$$

$$f(t) = 2h(t) - 2k(t) + g(t)$$

$$\therefore f(t) = 2e^{2t} - 2e^{t} + 1.$$

4

Consider the family of functions $f_1(t), f_2(t), \ldots$ that are defined by

$$f_n(t) = \frac{1}{n!}\frac{n}{T}\left(\frac{nt}{T}\right)^n e^{-nt/T} \text{ with } T > 0.$$

In order to determine the $n \to \infty$ limit of $f_n(t)$, first find the Laplace transform $F_n(s)$of $f_n(t)$, then evaluate $F_\infty(s) = \lim_{n\to\infty} F_n(s)$, and finally establish $f_\infty(t) = \lim_{n\to\infty} f_n(t)$.

Solution:

$$f_n(t) = \frac{1}{n!}\frac{n}{T}\left(\frac{nt}{T}\right)^n e^{-nt/T}$$

$$= \frac{1}{n!} t^n e^{-nt/T} \left(\frac{n}{T}\right)^{n+1},$$

$$f_n(t) \to F_n(s),$$

$$F_n(s) = \left(\frac{n}{T}\right)^{n+1} \int_0^\infty dt \left(e^{-st} \frac{1}{n!} t^n e^{-nt/T}\right)$$

$$= \left(\frac{1}{s+n/T}\right)^{n+1} \left(\frac{n}{T}\right)^{n+1}$$

$$= \left(\frac{n}{sT+n}\right)^{n+1}$$

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$$= \left(\frac{sT}{n} + 1\right)^{-n} \left(\frac{1}{\frac{sT}{n}+1}\right).$$

$$\lim_{n\to\infty} \left(\frac{sT}{n} + 1\right)^{-n} = e^{-sT},$$

$$\lim_{n\to\infty} \left(\frac{1}{\frac{sT}{n}+1}\right) = 1.$$

$$\therefore \lim_{n\to\infty} F_n(s) = \left(\lim_{n\to\infty} \left(\frac{sT}{n} + 1\right)^{-n}\right)\left(\lim_{n\to\infty} \left(\frac{sT}{n} + 1\right)^{-1}\right)$$

$$F_\infty = e^{-sT}.$$

$$\therefore f_\infty = \delta(t - T).$$