AY 2007/08, Sem 1, PC3274, Test 2

Question ...1/9....

Write answers on this side of the paper only.

Do not write on either margin

Solutions written up by Sim Yong Ming, Phoon Han Quan and Poon Wai Kit

1

The two 2×2 matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

are elements of a matrix group with just a few group elements. By considering $S^{-1}, R^{-1}, S^2, SR, RS, R^2, \ldots$, find the other group elements. Is the group abelian? If it isn't, identify the abelian subgroups.

Solution:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$R = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

$$S^{-1} = -1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$S^{-1} = S,$$

$$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I,$$

where **I** is the identity matrix,

$$R^{-1} = 2 \cdot \frac{1}{(-1)(-1) - (-\sqrt{3})(\sqrt{3})}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

$$= 2 \cdot \frac{1}{4}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

Question 2/9

Write answers on this side of the paper only.

Do not write on either margin

$$R^2 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1-3 & \sqrt{3}+\sqrt{3} \\ -\sqrt{3}-\sqrt{3} & -3+1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

$$R^2 = R^{-1},$$

notice that

$$RR^{-1} = I$$

$$= R^3,$$

Implying

$$(R^{-1})^2 = R.$$

$$SR = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix},$$

$$RS = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

$$R^{-1}S = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix},$$

$$R^{-1}S = SR,$$

Do not write on either margin

notice that, by multiplying the above with S on both side, we get

$$SR^{-1}SS = SSRS$$

$$SR^{-1}I = IRS$$

$$SR^{-1} = RS.$$

	R	R^2	S	RS	SR	I
R	R^2	I	RS	SR	S	R
R^2	I	R	SR	S	RS	R^2
S	SR	RS	I	R^2	R	S
RS	S	SR	R	I	R^2	RS
SR	RS	S	R^2	R	I	SR
I	R	R^2	S	RS	SR	I

From the table, we can deduce that R, R^{-1}, S, RS, SR, I are the group elements. The group is not abelian.

Subgroups which are abelian are $\{R, R^2, I\}$, $\{S, I\}$, $\{RS, I\}$, $\{SR, I\}$, and $\{I\}$.

2

The set G consists of all complex 2×2 matrices $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ whose matrix elements are restricted by the relations

$$|M_{11}|^2 = |M_{22}|^2 = 1 + |M_{21}|^2 = 1 + |M_{12}|^2, \qquad M_{21}^*M_{11} = M_{12}M_{22}^*,$$

Demonstrate that $M_{11}^*M_{12} = M_{22}M_{21}^*$, and then show that G is a group with matrix multiplication as the group composition law.

Solution:

Given

$$M_{21}^*M_{11} = M_{12}M_{22}^*,$$

$$M_{21}M_{11}^* = M_{12}^*M_{22},$$

so,

$$M_{21}^*M_{12}M_{21}M_{11}^* = M_{21}^*M_{12}M_{12}^*M_{22}^*$$

$$|M_{21}|^2 M_{12}M_{11}^* = |M_{12}|^2 M_{22}M_{21}^*$$

Do not write on either margin

$$\therefore M_{12}M_{11}^* = M_{22}M_{21}^*.$$

(i) To prove closure, there are two methods, as follows:

Method 1

$$|M_{11}|^2 = 1 + |M_{21}|^2$$

$$M_{11}^*M_{12} = \frac{M_{12}}{M_{11}} + \frac{M_{12}}{M_{11}}|M_{21}|^2,$$

$$|M_{22}|^2 = 1 + |M_{21}|^2$$

$$M_{22}M_{21}^* = \frac{M_{21}^*}{M_{22}^*} + \frac{M_{21}^*}{M_{22}^*}|M_{21}|^2,$$

$$A = \begin{pmatrix} M_{11}^* & -M_{21}^* \\ -M_{12}^* & M_{22}^* \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$AM = \begin{pmatrix} M_{11}^* & -M_{21}^* \\ -M_{12}^* & M_{22}^* \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11}^*M_{11} - M_{21}^*M_{21} & M_{11}^*M_{12} - M_{21}^*M_{22} \\ -M_{12}^*M_{11} + M_{22}^*M_{21} & -M_{12}^*M_{12} + M_{22}^*M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I.$$

Hence, A is the unique inverse of M.

$$\therefore M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $N \in G$,

$$(MN)^{-1} = N^{-1}M^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$