AY 2007/08, Sem I, PC3274, Exam

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$\boxed{1}$ The Euler Lagrange equation

$$2\left(\frac{d}{dx}\right)^2 f(x) = \frac{15}{2}\,\frac{f(x)^2}{\sqrt{x}}$$

is solved by $f(x) = x^{-3/2}$ which obeys the constraint $f(1)=1$, $f(\infty)=0$. The integral has the value

$$\int_1^\infty dx\,\left(\frac{9}{4}+\frac{5}{2}\right)x^{-5} = \frac{19}{16}\,.$$

$\boxed{2}$ In view of

$B^2: \; z \mapsto -z$

$B^3: \; z \mapsto -iz$

$B^4: \; z \mapsto z\,;$ $\qquad B^4 = E = BB^3 = B^2B^2 = B^3B$

$AB: \; z \mapsto -iz^*$ $\quad | \quad A^2 = E$

$AB^2: \; z \mapsto -z^*$

$AB^3: \; z \mapsto iz^*\,;$

$BA = AB^3$

$B^2A = BAB^3 = AB^2$

$ABA = A^2B^3 = B^3$

$BAB = AB^4 = A$

$BAB^2 = AB$

$B^3A = BAB^2 = AB$

the completed table is

Question 2/5

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	E	A	B	B^2	B^3	AB	AB^2	AB^3
E	E	A	B	B^2	B^3	AB	AB^2	AB^3
A	A	E	AB	AB^2	AB^3	B	B^2	B^3
B	B	AB^3	B^2	B^3	E	A	AB	AB^2
B^2	B^2	AB^2	B^3	E	B	AB^3	A	AB
B^3	B^3	AB	E	B	B^2	AB^2	AB^3	A
AB	AB	B^3	AB^2	AB^3	A	E	B	B^2
AB^2	AB^2	B^2	AB^3	A	AB	B^3	E	B
AB^3	AB^3	B	A	AB	AB^2	B^2	B^3	E

(b) Subgroups with two elements consist of

E and either A, or B^2, or AB, or AB^2, or AB^3;

Subgroups with four elements are

$\{E, B, B^2, B^3\}$, the cyclic group,

and $\{E, A, B^2, AB^2\}$, the vierergruppe,

and $\{E, B^2, AB, AB^3\}$, the vierergruppe again.

(c) All are abelian.

Question 3/5

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3 The Laplace transform $F(s) = \int_0^\infty dt\, e^{-st} f(t)$ obeys

$$F(s) + \frac{d}{ds}\left(sF(s) - f(0)\right) = 2F(s)^2$$

or $$s\frac{d}{ds}F(s) + 2F(s) = 2F(s)^2$$

and $$\int_0^\infty ds\, F(s) = \pi .$$

The differential equation has the general solution

$$F(s) = \frac{a}{a+s^2}$$

and we need $a > 0$ for the integral to converge. Since

$$\int_0^\infty ds\, \frac{a}{a+s^2} = \frac{\pi}{2}\sqrt{a}$$

we find $a = 4$, so that

$$F(s) = 2\,\frac{2}{s^2+2^2}$$

and $f(t) = 2\sin(2t)$.

Question 4/5

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$\boxed{4}$ (a) $$f(z)^2+1 = (\sinh\theta\cos\phi)^2+(\cos\phi)^2 - (\cosh\theta\sin\phi)^2+(\sin\phi)^2 + 2i\sinh\theta\cosh\theta\cos\phi\sin\phi$$
$$= (\cosh\theta\cos\phi + i\sinh\theta\sin\phi)^2 = z^2.$$

(b) In the limit $z \to |x_0|<1$ we have
$\theta \to 0$, $\cos\phi \to x_0$, $\sin\phi \to \pm\sqrt{1-x_0^2}$
so that
$$f(z_0+i\epsilon) \longrightarrow i\sqrt{1-x_0^2},$$
$$f(z_0-i\epsilon) \longrightarrow -i\sqrt{1-x_0^2},$$
and
$$f(z_0+i\epsilon) - f(z_0-i\epsilon) \longrightarrow 2i\sqrt{1-x_0^2}.$$

(c) Here $$dz = d\phi\,(-\cosh\theta\sin\phi + i\sinh\theta\cos\phi)$$
$$= i\,d\phi\, f(z)$$
so that
$$dz\, f(z) = i\,d\phi\, f(z)^2$$
$$= i\,d\phi\left[-\frac{1}{2}+\frac{1}{2}\cosh(2\theta)\cos(2\phi) + \frac{i}{2}\sinh(2\theta)\sin(2\phi)\right]$$
and $$\int_C dz\, f(z) = -i\pi$$
follows.

Question 5/5

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(d) We have

$$f(z) = z\left(1-\frac{1}{z^2}\right)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n} z\left(-\frac{1}{z^2}\right)^n$$

$$= \sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n z^{1-2n}$$

$$= z - \frac{1}{2z} + \frac{1}{8z^3} + \cdots$$

so that the residue is $-\frac{1}{2}$ and the integral in (c) is $(2\pi i)(-\frac{1}{2}) = -i\pi$, indeed.