AY 2007/08, Sem 1, PC4310, 1st Test

Question. 1/6

Write answers on this side of the paper only.

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$\boxed{1}$ (a) We have $UV|v_l\rangle = |v_{l+1}\rangle v_l$ and therefore

$$(UV)^N|v_l\rangle = |v_{l+N}\rangle v_l v_{l+1} \cdots v_{l+N-1}$$
$$= |v_l\rangle e^{i\frac{2\pi}{N}(l+(l+1)+\cdots+(l+N-1))}$$
$$= |v_l\rangle e^{i\frac{2\pi}{N}\frac{N}{2}(2l+N-1)}$$
$$= |v_l\rangle e^{i\pi(N-1)},$$

so that
$$(UV)^N = (-1)^{N-1}.$$

(b) Similarly, $U^mV^n|v_l\rangle = |v_{l+m}\rangle v_l^n$ implies

$$(U^mV^n)^N|v_l\rangle = |v_{l+mN}\rangle v_l^n v_{l+m}^n \cdots v_{l+m(N-1)}^n$$
$$= |v_l\rangle e^{i\frac{2\pi}{N}n\frac{N}{2}(2l+m(N-1))}$$
$$= |v_l\rangle e^{i\pi mn(N-1)}$$

and
$$(U^mV^n)^N = (-1)^{(N-1)mn}$$

follows.

(c) From $S|v_k\rangle = |u_k\rangle$ we get

$$S = \sum_k |u_k\rangle\langle v_k|.$$

And then

$$\langle u_k|S = \langle v_k|,$$

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as well as

$$S|u_k\rangle = \sum_\ell |u_\ell\rangle\langle v_\ell|u_k\rangle$$
$$= \sum_\ell |u_\ell\rangle \frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}\ell k}$$
$$= \sum_\ell |u_\ell\rangle \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}\ell(N-k)}$$
$$= \sum_\ell |u_\ell\rangle\langle u_\ell|v_{N-k}\rangle,$$

so that

$$S|u_k\rangle = |v_{N-k}\rangle,$$

and likewise $\langle v_k|S = \langle u_{N-k}|$.

(d) $$US = U\sum_k |u_k\rangle\langle v_k|$$
$$= \sum_k |u_k\rangle u_k \langle v_k|$$
$$= \sum_k |u_k\rangle v_k \langle v_k|$$
$$= \sum_k |u_k\rangle\langle v_k| V = SV.$$

(e) It is $S|v_k\rangle = |u_k\rangle$, $S|u_k\rangle = |v_{N-k}\rangle$,
$S|v_{N-k}\rangle = |u_{N-k}\rangle$, $S|u_{N-k}\rangle = |v_k\rangle$,
so that $S^4|v_k\rangle = |v_k\rangle$ for all k.
This states that $S^4 = 1$, so that S has period 4, except for $N=2$, where $S^2|v_1\rangle = |v_{2-1}\rangle = |v_1\rangle$, $S^2|v_2\rangle = |v_2\rangle$, and the period is 2.

$\boxed{2}$ (a) For $\underline{v=0}$, we have

$$i\hbar \frac{\partial}{\partial t_1} \langle x, t_1 | p, t_2 \rangle = \langle x, t_1 | (-F x(t_1)) | p, t_2 \rangle$$
$$= -F x \langle x, t_1 | p, t_2 \rangle$$

so that

$$\langle x, t_1 | p, t_2 \rangle = \langle x, t_2 | p, t_2 \rangle e^{\frac{i}{\hbar} F x (t_1 - t_2)}$$

or

$$\langle x, t_1 | p, t_2 \rangle = \frac{e^{ixp/\hbar}}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} F x T}$$

with $T = t_1 - t_2$.

(b) Since $\frac{\partial}{\partial v} H = |P(t)|$, we have

$$\frac{\partial}{\partial v} \langle x, t_1 | p, t_2 \rangle = -\frac{i}{\hbar} \langle x, t_1 | \int_{t_2}^{t_1} dt \, |P(t)| \, | p, t_2 \rangle$$

where $P(t)$ obeys

$$\frac{d}{dt} P = -\frac{\partial H}{\partial x} = F,$$

so that $P(t) = P(t_2) + F(t - t_2)$,

implying

$$\frac{\partial}{\partial v} \log \langle x, t_1 | p, t_2 \rangle = -\frac{i}{\hbar} \int_{t_2}^{t_1} dt \, |p + F(t - t_2)|$$

$p + F(t-t_2) = y$

$$= -\frac{i}{\hbar F} \int_p^{p+FT} dy \, |y| = -\frac{i}{2\hbar F} \left((p+FT)|p+FT| - p|p| \right).$$

It follows that

$$\langle x,t_1|p,t_2\rangle = \langle x,t_1|p,t_2\rangle\Big|_{v=0} e^{-\frac{iv}{2\hbar F}\left((p+FT)|p+FT|-p|p|\right)}$$

$$= \frac{e^{ixp/\hbar}}{\sqrt{2\pi\hbar}}\, e^{-\frac{iv}{2\hbar F}\left[(p+FT)|p+FT|-p|p|\right]+\frac{i}{\hbar}FxT}.$$

$\boxed{3}$(a)

$$\int_0^\pi \frac{d\vartheta}{\sin\vartheta}\langle\lambda|\vartheta\rangle\langle\vartheta|\lambda'\rangle$$

$$= \int_0^\pi \frac{d\vartheta}{\sin\vartheta}\frac{1}{2\pi}\left(\tan\frac{\vartheta}{2}\right)^{-i(\lambda-\lambda')}$$

substitute $e^{y} = \tan\frac{\vartheta}{2}$, $\vartheta=0:\ y=-\infty$, $\vartheta=\pi:\ y=\infty$

$$dy\, e^{y} = \frac{d\vartheta}{2(\cos\frac{\vartheta}{2})^2},\quad dy = \frac{d\vartheta}{2\tan\frac{\vartheta}{2}(\cos\frac{\vartheta}{2})^2} = \frac{d\vartheta}{\sin\vartheta}$$

to get

$$2\pi\int_0^\pi \frac{d\vartheta}{\sin\vartheta}\langle\lambda|\vartheta\rangle\langle\vartheta|\lambda'\rangle = \int_{-\infty}^{\infty} dy\, e^{-iy(\lambda-\lambda')}$$

$$= 2\pi\,\delta(\lambda-\lambda') = 2\pi\langle\lambda|\lambda'\rangle,$$

and

$$\int_0^\pi \frac{d\vartheta}{\sin\vartheta}|\vartheta\rangle\langle\vartheta| = 1 \quad \text{follows.}$$

Similarly,

$$\int_{-\infty}^{\infty} d\lambda\, \langle\vartheta|\lambda\rangle\langle\lambda|\vartheta'\rangle$$

$$= \int_{-\infty}^{\infty} d\lambda\, \frac{1}{2\pi}\left(\frac{\tan\frac{\vartheta}{2}}{\tan\frac{\vartheta'}{2}}\right)^{i\lambda}.$$

Question. 5/6

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$$= \delta\left(\log \frac{\tan\frac{\vartheta}{2}}{\tan\frac{\vartheta'}{2}}\right)$$

$$= \delta(\vartheta-\vartheta') \left| \left(\frac{\partial}{\partial\vartheta} \frac{\tan\frac{\vartheta}{2}}{\tan\frac{\vartheta'}{2}} \Bigg|_{\vartheta=\vartheta'} \right) \right|^{-1}$$

$$= \delta(\vartheta-\vartheta') \left| \frac{1}{2(\cos\frac{\vartheta}{2})^2 \tan\frac{\vartheta'}{2}} \Bigg|_{\vartheta=\vartheta'} \right|^{-1}$$

$$= \delta(\vartheta-\vartheta') \sin\vartheta = \langle\vartheta|\vartheta'\rangle,$$

implying

$$\int_{-\infty}^{\infty} d\lambda \, |\lambda\rangle\langle\lambda| = 1.$$

(b) Since $f(\Theta) = \int_0^\pi \frac{d\vartheta}{\sin\vartheta} |\vartheta\rangle f(\vartheta) \langle\vartheta|$ for any function of Θ, we have

$$\left(\tan\frac{\Theta}{2}\right)^{i\lambda'} |\lambda\rangle = \int_0^\pi \frac{d\vartheta}{\sin\vartheta} |\vartheta\rangle \underbrace{\left(\tan\frac{\vartheta}{2}\right)^{i(\lambda'+\lambda)}}_{\langle\vartheta|\lambda'+\lambda\rangle}$$

$$= |\lambda+\lambda'\rangle.$$

(c) We have $e^{i\mu\Lambda}\left(\tan\frac{\Theta}{2}\right)^{i\lambda}|\lambda'\rangle$

$$= e^{i\mu\Lambda} |\lambda+\lambda'\rangle = |\lambda+\lambda'\rangle e^{i\mu(\lambda+\lambda')}$$

$$= \left(\tan\frac{\Theta}{2}\right)^{i\lambda} |\lambda'\rangle e^{i\mu(\lambda+\lambda')}$$

$$= \left(\tan\frac{\Theta}{2}\right)^{i\lambda} e^{i\mu\Lambda} |\lambda'\rangle e^{i\mu\lambda}$$

which implies

$$e^{i\mu\Lambda}\left(\tan\frac{\theta}{2}\right)^{i\lambda} = e^{i\mu\lambda}\left(\tan\frac{\theta}{2}\right)^{i\lambda} e^{i\mu\Lambda}.$$

(d) Here we have

$$\langle\vartheta'| = \langle\vartheta| e^{i\mu\Lambda}$$
$$= e^{i\mu\lambda}\langle\vartheta|\left(\tan\frac{\theta}{2}\right)^{i\lambda} e^{i\mu\Lambda}\left(\tan\frac{\theta}{2}\right)^{-i\lambda}$$
$$= e^{i\mu\lambda}\left(\tan\frac{\vartheta}{2}\right)^{i\lambda}\left(\tan\frac{\vartheta'}{2}\right)^{-i\lambda}\langle\vartheta'|$$
$$= \underbrace{\left(e^{\mu}\left(\tan\frac{\vartheta}{2}\right)\left(\tan\frac{\vartheta'}{2}\right)^{-1}\right)^{i\lambda}}_{=1 \text{ for all } \lambda}\langle\vartheta'|$$

so that

$$\tan\frac{\vartheta'}{2} = e^{\mu}\tan\frac{\vartheta}{2}$$

or

$$\vartheta' = 2\arctan\left(e^{\mu}\tan\frac{\vartheta}{2}\right).$$