

Question... 1/6

Do not write on either margin

Write answers on this side of the paper only.

□(a) We have $UV|v_e\rangle = |v_{e+1}\rangle v_e$ and therefore

$$\begin{aligned} (UV)^N |v_e\rangle &= |v_{e+N}\rangle v_e v_{e+1} \cdots v_{e+N-1} \\ &= |v_e\rangle e^{i \frac{2\pi}{N} (1 + (1+1) + \cdots + (1+N-1))} \\ &= |v_e\rangle e^{i \frac{2\pi}{N} \frac{N}{2} (2N-1)} \\ &= |v_e\rangle e^{i\pi(N-1)}, \end{aligned}$$

so that

$$(UV)^N = (-1)^{N-1}.$$

(b) Similarly, $U^m V^n |v_e\rangle = |v_{e+m}\rangle v_e^n$ implies

$$\begin{aligned} (U^m V^n)^N |v_e\rangle &= |v_{e+mN}\rangle v_e^n v_{e+m}^n \cdots v_{e+m(N-1)}^n \\ &= |v_e\rangle e^{i \frac{2\pi}{N} n \frac{N}{2} (2N-1)} \\ &= |v_e\rangle e^{i\pi mn(N-1)} \end{aligned}$$

and

$$(U^m V^n)^N = (-1)^{(N-1)mn}$$

follows.

(c) From $S|v_k\rangle = |u_k\rangle$ we get

$$S = \sum_R |u_R\rangle \langle v_R|.$$

And then

$$\langle u_R | S = \langle v_R |,$$

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as well as

$$\begin{aligned}
 S|u_k\rangle &= \sum_e |u_e\rangle \langle v_e | u_k \rangle \\
 &= \sum_e |u_e\rangle \frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}ek} \\
 &= \sum_e |u_e\rangle \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}e(N-k)} \\
 &= \sum_e |u_e\rangle \langle u_e | v_{N-k} \rangle,
 \end{aligned}$$

so that

$$S|u_k\rangle = |v_{N-k}\rangle,$$

and likewise $\langle v_k | S = \langle u_{N-k} |$.

$$\begin{aligned}
 (d) \quad US &= U \sum_k |u_k\rangle \langle v_k | \\
 &= \sum_k |u_k\rangle u_k \langle v_k | \\
 &= \sum_k |u_k\rangle v_k \langle v_k | \\
 &= \sum_k |u_k\rangle \langle v_k | V = SV.
 \end{aligned}$$

$$\begin{aligned}
 (e) \quad \text{It is } S|v_k\rangle &= |u_k\rangle, \quad S|u_k\rangle = |v_{N-k}\rangle, \\
 S|v_{N-k}\rangle &= |u_{N-k}\rangle, \quad S|u_{N-k}\rangle = |v_k\rangle,
 \end{aligned}$$

so that $S^4|v_k\rangle = |v_k\rangle$ for all k . This states that $S^4 = 1$, so that S has period 4, except for $N=2$, where $S^2|v_1\rangle = |v_{2-1}\rangle = |v_1\rangle$, $S^2|v_2\rangle = |v_2\rangle$, and the period is 2.

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2 (a) For $v=0$, we have

$$i\hbar \frac{\partial}{\partial t_1} \langle x, t_1 | p, t_2 \rangle = \langle x, t_1 | (-Fx(t_1)) | p, t_2 \rangle$$

$$= -Fx \langle x, t_1 | p, t_2 \rangle$$

so that

$$\langle x, t_1 | p, t_2 \rangle = \langle x, t_2 | p, t_2 \rangle e^{\frac{i}{\hbar} Fx(t_1 - t_2)}$$

or

$$\langle x, t_1 | p, t_2 \rangle = \frac{e^{i x p / \hbar}}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} FxT}$$

with $T = t_1 - t_2$.

(b) Since $\frac{\partial}{\partial v} H = |P(t)|$, we have

$$\frac{\partial}{\partial v} \langle x, t_1 | p, t_2 \rangle = -\frac{i}{\hbar} \langle x, t_1 | \int_{t_2}^{t_1} dt |P(t)| | p, t_2 \rangle$$

where $P(t)$ obeys

$$\frac{d}{dt} P = -\frac{\partial H}{\partial x} = F,$$

so that

$$P(t) = P(t_2) + F(t - t_2),$$

implying

$$\frac{\partial}{\partial v} \log \langle x, t_1 | p, t_2 \rangle = -\frac{i}{\hbar} \int_{t_2}^{t_1} dt |p + F(t - t_2)|$$

$p + F(t - t_2) = y$

$$= -\frac{i}{\hbar F} \int_p^{p+FT} dy |y| = -\frac{i}{2\hbar F} ((p+FT)|p+FT| - p|p|).$$

It follows that

