

Question 1/4.....

Write answers on this side of the paper only.

Do not write on either margin

II We have

$$\begin{aligned}
 \overline{H_1(t)} &= e^{\frac{i}{\hbar} H_0 t} H_1 e^{-\frac{i}{\hbar} H_0 t} \\
 &= E_0 \cos(kx) e^{\frac{i}{\hbar} H_0 t} \times e^{-\frac{i}{\hbar} H_0 t} \\
 &= E_0 \cos(kx + kPt/M) \\
 &= \frac{1}{2} E_0 \left[e^{i kx + i kPt/M} + e^{-i kx - i kPt/M} \right] \\
 &= \frac{1}{2} E_0 \left[e^{i kx} e^{i kPt/M} e^{i \frac{1}{2} k^2 t/M} + e^{-i kx} e^{-i kPt/M} e^{i \frac{1}{2} k^2 t/M} \right] \\
 &= \frac{1}{2} E_0 \left[e^{i kx} e^{i k(P + \frac{1}{2} \hbar k) t/M} + e^{-i kx} e^{-i k(P - \frac{1}{2} \hbar k) t/M} \right]
 \end{aligned}$$

So that

$$\begin{aligned}
 S(T) &\cong 1 - \frac{i}{\hbar} \int_0^T dt \overline{H_1(t)} \\
 &= 1 - \frac{1}{2} E_0 \left[e^{i kx} \frac{e^{i k(P + \frac{1}{2} \hbar k) t/M} - 1}{\hbar k (P + \frac{1}{2} \hbar k) / M} - e^{-i kx} \frac{e^{-i k(P - \frac{1}{2} \hbar k) t/M} - 1}{\hbar k (P - \frac{1}{2} \hbar k) / M} \right]
 \end{aligned}$$

is the X,P ordered version of S(T) to first order in E₀.

Write answers on this side of the paper only.

[2] The instantaneous ground state of $H(x, p, t)$ has the position wave function

$$\psi_t(x) = \sqrt{k(t)} e^{-k(t)|x|},$$

so that we have

$$\psi_1(x) = \sqrt{k_1} e^{-k_1|x|} \quad \text{at } t=0$$

and

$$\psi_2(x) = \sqrt{k_2} e^{-k_2|x|} \quad \text{at } t=T.$$

In view of the very slow change of $k(t)$, the system evolves adiabatically and is in the instantaneous ground state at all times. Accordingly, the probability of having $\psi_1(x)$ at $t=T$ is

$$\begin{aligned} & \left| \int dx \psi_1(x)^* \psi_2(x) \right|^2 \\ &= \left| \sqrt{k_1 k_2} \int_{-\infty}^{\infty} dx e^{-(k_1 + k_2)|x|} \right|^2 \\ &= \left(\frac{2\sqrt{k_1 k_2}}{k_1 + k_2} \right)^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2} \\ &= 1 - \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2. \end{aligned}$$

Write answers on this side of the paper only.

3 The normalization $\int dx |\psi(x,t)|^2 = 1$ requires

$$1 = \int dx \left[|\alpha|^2 + |\beta|^2 x^2 + 2x \operatorname{Re}(\alpha^* \beta e^{-i\omega t}) \right] e^{-(x/a)^2}$$

$$= (a |\alpha|^2 + \frac{1}{2} a^3 |\beta|^2) \sqrt{\pi}.$$

The probability density $\rho(x,t)$ is (see above)

$$\rho(x,t) = \left[|\alpha|^2 + |\beta|^2 x^2 + 2x \operatorname{Re}(\alpha^* \beta e^{-i\omega t}) \right] e^{-(x/a)^2}$$

and the probability current density is

$$j(x,t) = \frac{\hbar}{M} \operatorname{Re} \left(\psi(x)^* \frac{\partial}{\partial x} \psi(x) \right)$$

$$= \operatorname{Re} \left(\frac{\hbar}{iM} \left(-\frac{2x}{a^2} \right) |\psi(x,t)|^2 + \frac{\hbar}{iM} (\alpha^* + \beta^* x e^{i\omega t}) \beta e^{-i\omega t} e^{-(x/a)^2} \right)$$

$$= \frac{\hbar}{M} \operatorname{Re} \left(\frac{1}{i} \alpha^* \beta e^{-i\omega t} \right) e^{-(x/a)^2}.$$

The continuity equation

$$0 = \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} j$$

$$= -2\omega x \operatorname{Re}(\alpha^* \beta e^{-i\omega t}) e^{-(x/a)^2} - \frac{\hbar}{M} \frac{2x}{a^2} \operatorname{Re}(\alpha^* \beta e^{-i\omega t}) e^{-(x/a)^2}$$

implies $\omega = \frac{\hbar}{Ma^2}$ or $Ma^2 \omega = \hbar$.

