$\boxed{1}$ We have

$$\overline{H_1}(t) = e^{\frac{i}{\hbar}H_0 t} H_1 e^{-\frac{i}{\hbar}H_0 t}$$

$$= E_0 \cos\left(k e^{\frac{i}{\hbar}H_0 t} X e^{-\frac{i}{\hbar}H_0 t}\right)$$

$$= E_0 \cos(kX + kPt/M)$$

$$= \frac{1}{2}E_0\left[e^{ikX + ikPt/M} + e^{-ikX - ikPt/M}\right]$$

$$= \frac{1}{2}E_0\left[e^{ikX} e^{ikPt/M} e^{i\hbar\frac{1}{2}k^2 t/M} + e^{-ikX} e^{-ikPt/M} e^{i\hbar\frac{1}{2}k^2 t/M}\right]$$

$$= \frac{1}{2}E_0\left[e^{ikX} e^{ik(P+\frac{1}{2}\hbar k)t/M} + e^{-ikX} e^{-ik(P-\frac{1}{2}\hbar k)t/M}\right]$$

so that

$$S(T) \cong 1 - \frac{i}{\hbar}\int_0^T dt\, \overline{H_1}(t)$$

$$= 1 - \frac{1}{2}E_0\left[e^{ikX}\frac{e^{ik(P+\frac{1}{2}\hbar k)t/M} - 1}{\hbar k(P+\frac{1}{2}\hbar k)/M} - e^{-ikX}\frac{e^{-ik(P-\frac{1}{2}\hbar k)t/M} - 1}{\hbar k(P-\frac{1}{2}\hbar k)/M}\right]$$

is the X,P ordered version of $S(T)$ to first order in E_0.

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$\boxed{2}$ The instantaneous ground state of $H(X, P, t)$ has the position wave function

$$\psi_t(x) = \sqrt{\kappa(t)}\, e^{-\kappa(t)|x|},$$

so that we have

$$\psi_1(x) = \sqrt{\kappa_1}\, e^{-\kappa_1|x|} \quad \text{at } t=0$$

and

$$\psi_2(x) = \sqrt{\kappa_2}\, e^{-\kappa_2|x|} \quad \text{at } t=T.$$

In view of the very slow change of $\kappa(t)$, the system evolves adiabatically and is in the instantaneous ground state at all times. Accordingly, the probability of having $\psi_1(x)$ at $t=T$ is

$$\left|\int dx\, \psi_1(x)^* \psi_2(x)\right|^2$$

$$= \left|\sqrt{\kappa_1\kappa_2}\int_{-\infty}^{\infty} dx\, e^{-(\kappa_1+\kappa_2)|x|}\right|^2$$

$$= \left(\frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2}\right)^2 = \frac{4\kappa_1\kappa_2}{(\kappa_1+\kappa_2)^2}$$

$$= 1 - \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 .$$

$\boxed{3}$ The normalization $\int dx\, |\psi(x,t)|^2 = 1$ requires

$$1 = \int dx \left[|\alpha|^2 + |\beta|^2 x^2 + 2x\, \mathrm{Re}(\alpha^* \beta e^{-i\omega t}) \right] e^{-(x/a)^2}$$

$$= \left(a|\alpha|^2 + \tfrac{1}{2} a^3 |\beta|^2 \right) \sqrt{\pi} \,.$$

The probability density $\rho(x,t)$ is (see above)

$$\rho(x,t) = \left[|\alpha|^2 + |\beta|^2 x^2 + 2x\, \mathrm{Re}(\alpha^* \beta e^{-i\omega t}) \right] e^{-(x/a)^2}$$

and the probability current density is

$$j(x,t) = \frac{\hbar}{M} \mathrm{Re}\left(\psi(x)^* \frac{1}{i} \frac{\partial}{\partial x} \psi(x) \right)$$

$$= \mathrm{Re}\Big(\frac{\hbar}{iM} \left(-\frac{2x}{a^2}\right) |\psi(x,t)|^2$$
$$+ \frac{\hbar}{iM} (\alpha^* + \beta^* e^{i\omega t}) \beta e^{-i\omega t} e^{-(\frac{x}{a})^2} \Big)$$

$$= \frac{\hbar}{M} \mathrm{Re}\left(\frac{1}{i} \alpha^* \beta e^{-i\omega t} \right) e^{-(x/a)^2} .$$

The continuity equation

$$0 = \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} j$$

$$= -2\omega x\, \mathrm{Re}(i\alpha^* \beta e^{-i\omega t}) e^{-(x/a)^2}$$
$$- \frac{\hbar}{M} \frac{2x}{a^2} \mathrm{Re}\left(\frac{1}{i} \alpha^* \beta e^{-i\omega t} \right) e^{-(x/a)^2}$$

implies $\quad \omega = \dfrac{\hbar}{Ma^2}$ or $\underline{M\omega a^2 = \hbar}$.

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$\boxed{4}$ Comparison with the Yukawa potential states

$$V(r) = V_0 e^{-r/a} = -\frac{\partial}{\partial \kappa}\left[\kappa \left(V_0 \frac{e^{-\kappa r}}{\kappa r}\right)\right]\Bigg|_{\kappa = 1/a}$$

so that

$$\int_0^\infty dr\, r\, V(r) \sin(qr)$$

$$= -\frac{\partial}{\partial \kappa}\left[\kappa \frac{V_0}{\kappa} \frac{q}{\kappa^2+q^2}\right]\Bigg|_{\kappa=\frac{1}{a}}$$

$$= \frac{2V_0 \kappa q}{(\kappa^2+q^2)^2}\Bigg|_{\kappa=1/a} = \frac{2V_0 a^3 q}{[1+(aq)^2]^2}$$

and

$$\frac{d\sigma}{d\Omega} = \left|\frac{2M}{\hbar^2 q} \frac{2V_0 a^3 q}{(1+a^2q^2)^2}\right|^2$$

$$= \left(\frac{4MV_0 a^3/\hbar^2}{(1+a^2q^2)^2}\right)^2 \xrightarrow[\text{or } \cancel{k}\to 0]{q\to 0} \left(\frac{4MV_0a^3}{\hbar^2}\right)^2$$

with $q = 2k \sin\frac{\theta}{2}$ as usual.

Since $d\Omega = 2\pi\, d\theta \sin\theta = \frac{\pi}{k^2} dq^2$, the total cross section is

$$\sigma = \frac{\pi}{k^2}\left(\frac{4MV_0a^3}{\hbar^2}\right)^2 \int_0^{(2k)^2} \frac{dy}{(1+a^2y)^4}$$

$$= \frac{\pi}{3(ka)^2}\left(\frac{4MV_0a^3}{\hbar^2}\right)^2 \left(1 - \frac{1}{(1+4k^2a^2)^3}\right).$$