Problem 1 (30 marks)
Mass M moves along the x axis. The Hamilton operator

$$H(X,P) = H_0 + H_1 \quad \text{with} \quad H_0 = \frac{1}{2M}P^2 \quad \text{and} \quad H_1 = E_0 \cos(kX)$$

governs the evolution, whereby energy E_0 and wave number k are positive constants. Find the scattering operator

$$S(T) = \mathrm{e}^{\mathrm{i}H_0T/\hbar}\mathrm{e}^{-\mathrm{i}HT/\hbar}$$

to first order in E_0. It suffices to state the X,P-ordered version of $S(T)$.

Problem 2 (25 marks)
The one-dimensional dynamics of a particle of mass M is controlled by the Hamilton operator

$$H(X,P,t) = \frac{1}{2M}P^2 - \frac{\hbar^2\kappa(t)}{M}\delta(X)$$

which has a parametric time dependence that originates in the time-dependent parameter $\kappa(t) > 0$. Before $t = 0$, $\kappa(t)$ has the constant value κ_1; after $t = T > 0$, it has the constant value κ_2. Between $t = 0$ and $t = T$, the value of $\kappa(t)$ changes very slowly. Given that the system is in its ground state at $t = 0$, what is the probability that it is still in the same state at $t = T$?

Problem 3 (20 marks)
Mass M is moving along the x axis, with the Hamilton operator

$$H(X,P,t) = \frac{1}{2M}P^2 + V(X,t)$$

governing the evolution, whereby $V(X,t)$ is an unknown potential energy that could have a parametric time dependence. The position wave function is of the form

$$\psi(x,t) = \left(\alpha + \beta x\,\mathrm{e}^{-\mathrm{i}\omega t}\right)\mathrm{e}^{-\frac{1}{2}(x/a)^2}$$

with constant parameters α, β, ω, and $a > 0$. Which relation among these parameters is implied by the normalization of $\psi(x,t)$? How are ω and a related to each other? — Hint: Remember the continuity equation.

Problem 4 (25 marks)
A particle with mass M and energy $(\hbar k)^2/(2M)$ is scattered by the spherically symmetric potential

$$V(r) = V_0\,\mathrm{e}^{-r/a}$$

with $a > 0$. Find the differential scattering cross section $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ and the total cross section σ in first-order Born approximation. What is the $k \to 0$ limit of $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$?