

Write answers on this side of the paper only.

□ (a) We recall that $A = \frac{1}{\sqrt{2kM\omega}} (M\omega X + iP)$ and

note that $\frac{d}{dt} X = \frac{1}{M} P$, $\frac{d}{dt} P = -M\omega^2 X + F(t)$

imply $\frac{d}{dt} A(t) = -i\omega A(t) + \frac{i}{\sqrt{2kM\omega}} F(t)$

or $\frac{d}{dt} [e^{i\omega t} A(t)] = \frac{i}{\sqrt{2kM\omega}} e^{i\omega t} F(t)$,

so that

$$e^{i\omega T} A(T) = A(0) + \frac{i}{\sqrt{2kM\omega}} \int_0^T dt e^{i\omega t} F(t).$$

Here we have

$$\int_0^T dt e^{i\omega t} F(t) = \int_{-\infty}^{\infty} dt e^{i\omega t} F(t) = f(\omega)$$

and arrive at

$$A(T) = e^{-i\omega T} \left(A(0) + \frac{i f(\omega)}{\sqrt{2kM\omega}} \right)$$

as well as (take the adjoint)

$$A^\dagger(T) = e^{i\omega T} \left(A^\dagger(0) - \frac{i f(\omega)^*}{\sqrt{2kM\omega}} \right).$$

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$$\begin{aligned} \text{II}(b) \text{ From } \langle n, T | &= \frac{1}{\sqrt{n!}} \langle 0, T | A(T)^n \\ &= \frac{1}{\sqrt{n!}} \langle 0, T | \left[e^{-i\omega T} \left(A(0) + \frac{i f(\omega)}{\sqrt{2\hbar M \omega}} \right) \right]^n \end{aligned}$$

and $A(0) |0, 0\rangle = 0$, we get

$$\langle n, T | 0, 0 \rangle = \frac{1}{\sqrt{n!}} \left(e^{-i\omega T} \frac{i f(\omega)}{\sqrt{2\hbar M \omega}} \right)^n \langle 0, T | 0, 0 \rangle.$$

$$\begin{aligned} \text{II}(c) \text{ We have } P_n(T) &= \frac{1}{n!} \left(\frac{|f(\omega)|^2}{2\hbar M \omega} \right)^n P_0(T) \\ &= \frac{1}{n!} \nu^n P_0(T) \end{aligned}$$

with $\nu = \frac{1}{2\hbar M \omega} |f(\omega)|^2$, so that

$$1 = \sum_{n=0}^{\infty} P_n(T) = e^{\nu} P_0(T), \text{ which implies}$$

first $P_0(T) = e^{-\nu}$ and then

$$P_n(T) = \frac{\nu^n}{n!} e^{-\nu}.$$

II(d) Yes, this is possible if $F(t)$ is such that

$$f(\omega) = 0,$$

that is: $F(t)$ has no Fourier component for the frequency of the harmonic oscillator.

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2(a) When there is no θ dependence, we have

$$\frac{d\sigma}{d\Omega} = |f|^2 \quad \text{and} \quad \sigma = 4\pi |f|^2 = \frac{4\pi}{k} \operatorname{Im} f$$

or

$$f^* f = \frac{1}{2ik} (f - f^*),$$

so that $\frac{1}{2i} \left(\frac{1}{f} - \frac{1}{f^*} \right) = -k,$

which implies $\frac{1}{f} = -ik - \frac{1}{b(k)}$

with some real function $b(k)$. Accordingly,

$$f = - \frac{b(k)}{1 + ikb(k)}, \quad \text{indeed.}$$

2(b) For s-wave scattering, one has $f = \frac{1}{k} e^{i\delta_0} \sin \delta_0$

or

$$\frac{1}{kf} = \frac{e^{-i\delta_0}}{\sin \delta_0} = \cot \delta_0 - i,$$

to be compared with $\frac{1}{kf} = -i - \frac{1}{kb(k)},$

from which

$$b(k) = -\frac{1}{k} \tan \delta_0$$

follows immediately.

2(c) From (3.6.8) in the notes, we get

$$b_0 = -\frac{1}{k} \tan \delta_0 \Big|_{k \rightarrow 0} = a - \frac{1}{k} \tanh(ka).$$

