AY 2007/08, Sem 1, PC4130, Exam

Question. . . 1/5 . . .

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[1] (a) We recall that $A = \frac{1}{\sqrt{2\hbar M\omega}}(M\omega X + iP)$ and

note that $\frac{d}{dt}X = \frac{1}{M}P$, $\frac{d}{dt}P = -M\omega^2 X + F(t)$

imply

$$\frac{d}{dt}A(t) = -i\omega A(t) + \frac{i}{\sqrt{2\hbar M\omega}}F(t)$$

or

$$\frac{d}{dt}\left[e^{i\omega t}A(t)\right] = \frac{i}{\sqrt{2\hbar M\omega}}e^{i\omega t}F(t),$$

so that

$$e^{i\omega T}A(T) = A(0) + \frac{i}{\sqrt{2\hbar M\omega}}\int_0^T dt\, e^{i\omega t}F(t).$$

Here we have

$$\int_0^T dt\, e^{i\omega t}F(t) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}F(t) = f(\omega)$$

and arrive at

$$A(T) = e^{-i\omega T}\left(A(0) + \frac{i f(\omega)}{\sqrt{2\hbar M\omega}}\right)$$

as well as (take the adjoint)

$$A^\dagger(T) = e^{i\omega T}\left(A^\dagger(0) - \frac{i f(\omega)^*}{\sqrt{2\hbar M\omega}}\right).$$

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$\boxed{\text{I}}$(b) From $\langle n,T| = \frac{1}{\sqrt{n!}} \langle 0,T| \, A(T)^n$

$$= \frac{1}{\sqrt{n!}} \langle 0,T| \left[e^{-i\omega T} \left(A(0) + \frac{i f(\omega)}{\sqrt{2\hbar M\omega}} \right) \right]^n$$

and $A(0)\,|0,0\rangle = 0$, we get

$$\langle n,T|0,0\rangle = \frac{1}{\sqrt{n!}} \left(e^{-i\omega T} \frac{i f(\omega)}{\sqrt{2\hbar M\omega}} \right)^n \langle 0,T|0,0\rangle .$$

$\boxed{\text{I}}$(c) We have $p_n(T) = \frac{1}{n!} \left(\frac{|f(\omega)|^2}{2\hbar M\omega} \right)^n p_0(T)$

$$= \frac{1}{n!} \nu^n p_0(T)$$

with $\nu = \frac{1}{2\hbar M\omega} |f(\omega)|^2$, so that

$1 = \sum_{n=0}^{\infty} p_n(T) = e^{\nu} p_0(T)$, which implies

first $p_0(T) = e^{-\nu}$ and then

$$p_n(T) = \frac{\nu^n}{n!} e^{-\nu} .$$

$\boxed{\text{I}}$(d) Yes, this is possible if $F(t)$ is such that

$$f(\omega) = 0,$$

that is: $F(t)$ has no Fourier component for the frequency of the harmonic oscillator.

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$\boxed{2}$ (a) When there is no θ dependence, we have

$$\frac{d\sigma}{d\Omega} = |f|^2 \quad \text{and} \quad \sigma = 4\pi\, |f|^2 = \frac{4\pi}{k}\, \mathrm{Im}\, f$$

optical theorem

or

$$f^* f = \frac{1}{2ik}\left(f - f^*\right),$$

so that $\frac{1}{2i}\left(\frac{1}{f} - \frac{1}{f^*}\right) = -k$,

which implies $\frac{1}{f} = -ik - \frac{1}{b(k)}$

with some <u>real</u> function $b(k)$. Accordingly,

$$f = -\frac{b(k)}{1 + ik\, b(k)}, \quad \text{indeed.}$$

$\boxed{2}$ (b) For s-wave scattering, one has $f = \frac{1}{k}\, e^{i\delta_0} \sin\delta_0$

or

$$\frac{1}{kf} = \frac{e^{-i\delta_0}}{\sin\delta_0} = \cot\delta_0 - i,$$

to be compared with $\frac{1}{kf} = -i - \frac{1}{k\,b(k)}$,

from which

$$b(k) = -\frac{1}{k}\tan\delta_0$$

follows immediately.

$\boxed{2}$ (c) From (3.6.8) in the notes, we get

$$b_0 = -\frac{1}{k}\tan\delta_0\Big|_{k\to 0} = a - \frac{1}{\kappa}\tanh(\kappa a).$$

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$\boxed{2}$ (d) Since $\left.\frac{d\sigma}{d\Omega}\right|_{k\to 0} = |f|^2\Big|_{k\to 0} = |b_0|^2$ and

$$\left.\frac{d\sigma}{d\Omega}\right|_{k\to 0} = \left(\frac{V_0/K}{\frac{(\hbar K)^2}{2M}}\right)^2 \qquad \text{(see (3.3.27) in the notes)}$$

we get $|b_0| = \dfrac{2M|V_0|}{\hbar^2 K^3}$.

$\boxed{3}$ (a) We have $\langle \tfrac{1}{2};\nu|\tfrac{1}{2};\nu'\rangle = \frac{1}{3}\left(1+q^{\nu'-\nu}+q^{\nu-\nu'}\right)$

$$= \frac{1}{3}\begin{Bmatrix} 1+1+1 & \text{for } \nu=\nu' \\ 1+q+q^2 & \text{for } \nu\neq\nu' \end{Bmatrix} = \delta_{\nu\nu'},$$

which shows the orthonormality, and

$$S_z|{+}{+}{-}\rangle = (S_{1z}+S_{2z}+S_{3z})|{+}{+}{-}\rangle$$
$$= |{+}{+}{-}\rangle\left(\tfrac{1}{2}\hbar+\tfrac{1}{2}\hbar-\tfrac{1}{2}\hbar\right) = |{+}{+}{-}\rangle\tfrac{1}{2}\hbar$$

and likewise for $|{+}{-}{+}\rangle$ and $|{-}{+}{+}\rangle$

shows that $S_z|\tfrac{1}{2};\nu\rangle = |\tfrac{1}{2};\nu\rangle\tfrac{1}{2}\hbar$.

$\boxed{3}$ (b) For a single spin-$\frac{1}{2}$ system we have

$S_+|+\rangle = 0$, $S_+|-\rangle = |+\rangle\hbar$,

$S_-|+\rangle = |-\rangle\hbar$, and $S_-|-\rangle = 0$.

For the system of three spin-$\frac{1}{2}$ systems, we thus have

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$$S_+|\tfrac{1}{2};\nu\rangle = \frac{1}{\sqrt{3}}\ |{+}{+}{+}\rangle\, \hbar\,(1+q^{\nu}+q^{-\nu})$$

$$= \begin{cases} |{+}{+}{+}\rangle\, \hbar\sqrt{3} & \text{for } \nu=0, \\ 0 & \text{for } \nu=\pm 1, \end{cases}$$

so that $|\tfrac{3}{2};0\rangle = |{+}{+}{+}\rangle$

and there are no states $|\tfrac{3}{2};\pm 1\rangle$.

Further $$S_-|\tfrac{1}{2};\nu\rangle = \frac{1}{\sqrt{3}}\Big(|{+}{-}{-}\rangle\, \hbar\,(1+q^{\nu}) + |{-}{+}{-}\rangle\, \hbar\,(1+q^{-\nu}) + |{-}{-}{+}\rangle\, \hbar\,(q^{\nu}+q^{-\nu})\Big)$$

$$= \begin{cases} \dfrac{2\hbar}{\sqrt{3}}\big(|{+}{-}{-}\rangle+|{-}{+}{-}\rangle+|{-}{-}{+}\rangle\big) & \text{for } \nu=0 \\ -\dfrac{\hbar}{\sqrt{3}}\big(|{+}{-}{-}\rangle q^{-\nu}+|{-}{+}{-}\rangle q^{\nu}+|{-}{-}{+}\rangle\big) & \text{for } \nu=\pm 1 \end{cases}$$

so that
$$|-\tfrac{1}{2};\nu\rangle = \frac{(-1)^{\nu}}{\sqrt{3}}\big(|{+}{-}{-}\rangle q^{-\nu}+|{-}{+}{-}\rangle q^{\nu}+|{-}{-}{+}\rangle\big).$$

Finally $$S_-|-\tfrac{1}{2};\nu\rangle = \begin{cases} |{-}{-}{-}\rangle\,\hbar\sqrt{3} & \text{for } \nu=0 \\ 0 & \text{for } \nu=\pm 1 \end{cases}$$

so that $|-\tfrac{3}{2};0\rangle = |{-}{-}{-}\rangle$

and there are no states $|-\tfrac{3}{2};\pm 1\rangle$.

$\boxed{3}$ (c) Clearly, the 4 states with $\nu=0$ have total spin $\frac{3}{2}$, the two pairs of states with $\nu=1$ and $\nu=-1$ have total spin $\frac{1}{2}$, so that the eigenvalue of $\vec{S}^2$ is $\frac{3}{2}(\frac{3}{2}+1)\hbar^2 = \frac{15}{4}\hbar^2$ for $\nu=0$ and $\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$ for $\nu=\pm 1$.