1. **Time-dependent Perturbation (35=10+10+10+5 marks)**
A harmonic oscillator (mass M, angular frequency ω, position operator X, momentum operator P) is driven by a time-dependent force $F(t)$, so that the Hamilton operator is given by

$$H = \frac{1}{2M}P^2 + \frac{1}{2}M\omega^2 X^2 - F(t)X\,,$$

whereby the force is only acting during the interval $0 < t < T$, that is

$$F(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega' t} f(\omega') = \begin{cases} 0 \text{ for } t < 0 \text{ and } t > T\,, \\ \text{anything for } 0 < t < T\,. \end{cases}$$

(a) State the equations of motion for the ladder operators $A(t)$ and $A^\dagger(t)$ and solve them to express the operators at time $t = T$ in terms of those at time $t = 0$.

(b) Next, establish that the transition amplitudes $\langle n, t = T | n = 0, t = 0\rangle$ from the oscillator ground state at $t = 0$ to its nth excited state at $t = T$ are of the form

$$\langle n, T | 0, 0\rangle \propto \langle 0, T | 0, 0\rangle f(\omega)^n\,.$$

(c) Determine the ground-state persistence probability $p_0(T) = |\langle 0, T | 0, 0\rangle|^2$ by a normalization argument, and then state explicitly the transition probabilities $p_n(T) = |\langle n, T | 0, 0\rangle|^2$ for arbitrary n.

(d) Is it possible that a nonzero force has no net effect, that is: can you have $p_0(T) = 1$ although $F(t) \neq 0$ for $0 < t < T$?

2. Scattering (35=10+10+10+5 marks)

At very low energies, scattering is s-wave scattering only and the scattering amplitude $f(\vec{k}',\vec{k})$ does not depend on the scattering angle θ.

(a) Show that

$$f = -\frac{b(k)}{1+\mathrm{i}kb(k)} \qquad \text{with } b(k) \text{ real}$$

under these circumstances. — Hint: Remember the Optical Theorem.

(b) How is $b(k)$ related to the s-wave scattering phase shift $\delta_0(k)$?

(c) The *scattering length* b_0 is the $k \to 0$ limit of $b(k)$. Determine b_0 for the repulsive hard-sphere potential

$$V(r) = \begin{cases} \dfrac{(\hbar\kappa)^2}{2M} & \text{for } r < a\,, \\ 0 & \text{for } r > a\,, \end{cases}$$

with $\kappa > 0$.

(d) What is $|b_0|$ for the Yukawa potential $V(r) = V_0 \dfrac{\mathrm{e}^{-\kappa r}}{\kappa r}$ $(\kappa > 0)$ in Born approximation?

3. Angular Momentum (30=10+15+5 marks)

Consider three spin-$\frac{1}{2}$ systems with individual spin vector operators $\vec{S}_1$, $\vec{S}_2$, and $\vec{S}_3$, and total spin vector operator $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$. We denote by $|+-+\rangle$, for instance, the common eigenket of S_{1z}, S_{2z}, and S_{3z} with respective eigenvalues $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, and $\frac{1}{2}\hbar$.

(a) Show that the three kets $|\frac{1}{2};\nu\rangle$ that are defined, for $\nu = 0, \pm 1$, by

$$|\tfrac{1}{2};\nu\rangle = \frac{1}{\sqrt{3}}\Big(|++-\rangle + |+-+\rangle q^{\nu} + |-++\rangle q^{-\nu}\Big) \qquad \text{with } q = \mathrm{e}^{\mathrm{i}2\pi/3}$$

are properly normalized, pairwise orthogonal eigenkets of S_z with eigenvalue $\frac{1}{2}\hbar$. — Note: Observe that $1 + q + q^2 = 0$ and $q^2 = q^* = q^{-1}$.

(b) Construct the corresponding kets $|m;\nu\rangle$, for which $S_z|m;\nu\rangle = |m;\nu\rangle m\hbar$, by applying the ladder operators $S_\pm = S_x \pm \mathrm{i}S_y$. State explicitly what you get for $|-\frac{1}{2};\nu\rangle$ and, if they exist, $|\pm\frac{3}{2};\nu\rangle$.

(c) What is the eigenvalue of $\vec{S}^2$ for the set of kets with $\nu = 0$; what is it for the sets with $\nu = \pm 1$?