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1 (a) Since $\begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$ and $\begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}$ are 4-vector fields, their Lorentz scalar product,

$$(-c\rho, \vec{j}^T)\begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} = -c\left[\rho\Phi - \frac{1}{c}\vec{j}\cdot\vec{A}\right],$$

is a <u>4-scalar field</u>.

(b) The gauge transformation $\vec{A} \to \vec{A} - \vec{\nabla}\lambda$, $\Phi \to \Phi + \frac{1}{c}\frac{\partial}{\partial t}\lambda$ adds

$$\int(d\vec{r})\left(\rho\frac{1}{c}\frac{\partial}{\partial t}\lambda + \frac{1}{c}\vec{j}\cdot\vec{\nabla}\lambda\right)$$

$$= \underbrace{\int(d\vec{r})\, \vec{\nabla}\cdot\left(\frac{1}{c}\vec{j}\lambda\right)}_{=0 \text{ (surface integral)}} + \int(d\vec{r})\Big(\rho\frac{1}{c}\frac{\partial}{\partial t}\lambda - \frac{1}{c}\lambda\underbrace{\vec{\nabla}\cdot\vec{j}}_{=-\frac{\partial}{\partial t}\rho}\Big)$$

$$= \int(d\vec{r})\,\frac{1}{c}\frac{\partial}{\partial t}(\lambda\rho) = \frac{1}{c}\frac{d}{dt}\int(d\vec{r})\,\lambda\rho,$$

which is a total time derivative, indeed.

2 (a) With the mirror moving in the z direction, and the wave vectors of the incoming and reflected light in the x, z plane, we have

$\begin{pmatrix} \gamma c \\ 0 \\ 0 \\ \gamma v \end{pmatrix}$ for the 4-velocity of the mirror,

and $\frac{\omega}{c}\begin{pmatrix} 1 \\ \sin\vartheta \\ 0 \\ -\cos\vartheta \end{pmatrix}$, $\frac{\omega'}{c'}\begin{pmatrix} 1 \\ \sin\vartheta' \\ 0 \\ \cos\vartheta' \end{pmatrix}$ for the

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4-wave vector of the plane waves. In the rest-frame of the mirror we would have

$\begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}$ for the 4-velocity, and $\frac{\omega_0}{c}\begin{pmatrix} 1 \\ \sin\vartheta_0 \\ 0 \\ \mp\cos\vartheta_0 \end{pmatrix}$

for the two 4-wave vectors.

The Lorentz-transformation into the rest frame of the mirror is:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \longrightarrow \gamma \begin{pmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1/\gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ -v/c & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

as verified by

$$\begin{pmatrix} \gamma c \\ 0 \\ 0 \\ \gamma v \end{pmatrix} \longrightarrow c\gamma^2 \begin{pmatrix} 1-v^2/c^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So,

$$\frac{\omega_0}{c}\begin{pmatrix} 1 \\ \sin\vartheta_0 \\ 0 \\ -\cos\vartheta_0 \end{pmatrix} = \gamma \begin{pmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1/\gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ -v/c & 0 & 0 & 1 \end{pmatrix} \frac{\omega}{c}\begin{pmatrix} 1 \\ \sin\vartheta \\ 0 \\ -\cos\vartheta \end{pmatrix}$$

$$= \gamma\frac{\omega}{c}\begin{pmatrix} 1+\frac{v}{c}\cos\vartheta \\ \frac{1}{\gamma}\sin\vartheta \\ 0 \\ -\cos\vartheta - v/c \end{pmatrix}$$

and, likewise,

$$\frac{\omega_0}{c}\begin{pmatrix} 1 \\ \sin\vartheta_0 \\ 0 \\ \cos\vartheta_0 \end{pmatrix} = \gamma\frac{\omega'}{c}\begin{pmatrix} 1-\frac{v}{c}\cos\vartheta' \\ \frac{1}{\gamma}\sin\vartheta' \\ 0 \\ \cos\vartheta' - v/c \end{pmatrix},$$

and upon equating the time components and

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the z components, we have

$$\omega\left(1+\frac{v}{c}\cos\vartheta\right) = \omega'\left(1-\frac{v}{c}\cos\vartheta'\right),$$
$$\omega\left(\cos\vartheta+\frac{v}{c}\right) = \omega'\left(\cos\vartheta'-\frac{v}{c}\right).$$

These imply

$$\boxed{\omega' = \omega\,\gamma^2\left[1+\frac{2v}{c}\cos\vartheta+\left(\frac{v}{c}\right)^2\right] \quad \text{and} \quad \cos\vartheta' = \frac{\cos\vartheta+u}{1+u\cos\vartheta} \quad \text{with } u=\frac{2v/c}{1+(v/c)^2}.}$$

(b) When $v/c\to 1$, then $\underline{\omega'\to\infty}$ and $\cos\vartheta'\to 1$, or $\underline{\vartheta'\to 0}$.

(c) When $v=-c\cos\vartheta$, then the mirror recedes at the speed by which the incoming light approaches the mirror, and there should then be no reflection, so that $\omega'=\omega$ and $\vartheta'=\pi-\vartheta$ is expected. This is also what the equations tell:

$$\omega' = \omega\,\frac{1}{1-(\cos\vartheta)^2}\left(1-2(\cos\vartheta)^2+(\cos\vartheta)^2\right) = \omega,$$

$$\cos\vartheta' = \frac{\cos\vartheta-\dfrac{2\cos\vartheta}{1+(\cos\vartheta)^2}}{1-\dfrac{2(\cos\vartheta)^2}{1+(\cos\vartheta)^2}} = -\cos\vartheta = \cos(\pi-\vartheta),$$

indeed.

(d) For $\vartheta' = \frac{1}{2}\pi$, we have $\cos\vartheta' = 0$, and this requires $u = -\cos\vartheta$, or

$$1 + 2\,\frac{v/c}{\cos\vartheta} + (v/c)^2 = 0,$$

solved by
$$\frac{v}{c} = -\frac{1}{\cos\vartheta} \pm \sqrt{\left(\frac{1}{\cos\vartheta}\right)^2 - 1}$$

$$= -\frac{1 \mp \sin\vartheta}{\cos\vartheta} = -\frac{1 \mp \sin\vartheta}{\sqrt{(1 \mp \sin\vartheta)(1 \pm \sin\vartheta)}}$$

$$= -\sqrt{\frac{1 \mp \sin\vartheta}{1 \pm \sin\vartheta}},$$

where only the upper sign is physically possible, so that

$$v = -\sqrt{\frac{1 - \sin\vartheta}{1 + \sin\vartheta}}\; c = -\frac{1 - \sin\vartheta}{\cos\vartheta}\, c$$

is the answer.

$\boxed{3}$ (a) Since $\vec{\nabla}\cdot\vec{j}(\vec{r},t) = -\frac{\partial}{\partial t}\rho(\vec{r},t) = \dot{\vec{d}}(t)\cdot\vec{\nabla}\delta(\vec{r})$,

we have $\vec{j}(\vec{r},t) = \dot{\vec{d}}(t)\,\delta(\vec{r}) + [\text{a curl, possibly}]$,

but as parts (b) and (c) show, there is actually no need to add a curl of something.

(b) Without the extra curl:

$$\vec{\mu}(t) = \frac{1}{2c}\int(d\vec{r})\;\vec{r}\times\dot{\vec{d}}(t)\,\delta(\vec{r}) = 0.$$

(c) Without the extra curl,

$$\int (d\vec{r})\, \vec{j}(\vec{r},t) = \dot{\vec{d}}(t)$$, as it should be, and the extra curl term would not matter here.

(d) It is

$$\vec{A}(\vec{r},t) = \int (d\vec{r}\,') \int dt' \, \frac{\delta(t-t'-\frac{1}{c}|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|} \frac{1}{c}\dot{\vec{d}}(t')\,\delta(\vec{r}\,')$$

$$= \frac{1}{rc}\dot{\vec{d}}(t-r/c).$$

$\boxed{4}$ Larmor says: $\frac{dP}{d\Omega}(t) = \frac{1}{4\pi c^3}\left|\vec{n}\times\ddot{\vec{d}}(t)\right|^2$

where $\ddot{\vec{d}}(t) = \vec{\omega}\times(\vec{\omega}\times\vec{d}(t)) = -\omega^2\vec{d}(t)$,

and
$$\left|\vec{n}\times\vec{d}(t)\right|^2 = \vec{d}(t)^2 - (\vec{n}\cdot\vec{d}(t))^2$$

$$\xrightarrow[\text{over one period}]{\text{averaged}} d^2 - \frac{1}{2}(\sin\theta)^2 d^2,$$

so that
$$\frac{dP}{d\Omega} = \frac{\omega^4 d^2}{4\pi c^3}\left[1-\frac{1}{2}(\sin\theta)^2\right],$$

and the total power is

$$P = \frac{\omega^4 d^2}{c^3}\left(1-\frac{1}{2}\cdot\frac{2}{3}\right) = \frac{2}{3}\frac{\omega^4 d^2}{c^3},$$

which we get by integration over the solid angle of 4π, or alternatively from

$$\frac{dP}{d\Omega} = \frac{2}{3c^3}\left|\ddot{d}\right|^2.$$