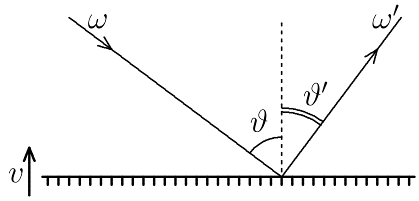
Problem 1 (15=6+9 marks)
As usual, $\rho(\vec{r},t)$ and $\vec{j}(\vec{r},t)$ denote the electric charge density and the electric current density, respectively, and $\Phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ are the scalar potential and the vector potential.

(a) How does $\rho\Phi - \dfrac{1}{c}\vec{j}\cdot\vec{A}$ transform under Lorentz transformations — like a 4-scalar field, like the time-like component of a 4-vector field, or like something else?

(b) Show that a gauge transformation amounts to adding a total time derivative to $\displaystyle\int(\mathrm{d}\vec{r})\left(\rho\Phi - \frac{1}{c}\vec{j}\cdot\vec{A}\right)$.

Problem 2 (45=20+7+8+10 marks)
A monochromatic plane light wave of (angular) frequency ω is incident on a plane mirror under normal angle ϑ. The mirror moves with constant normal velocity v. The reflected wave has frequency ω' and normal angle ϑ'.
For $v = 0$, we have $\omega' = \omega$ and $\vartheta' = \vartheta$.



(a) With the convention that the mirror moves *toward* the incoming wave for $v > 0$, as indicated in the figure, express ω' and $\cos\vartheta'$ in terms of v, ω, and $\cos\vartheta$.

(b) What do you get for ω' and ϑ' in the limit $v \to c$?

(c) Why do you expect $\omega' = \omega$ and $\vartheta' = \pi - \vartheta$ when $v = -c\cos\vartheta$? Verify that your expressions confirm this expectation.

(d) For which (negative) value of v is $\vartheta' = \frac{1}{2}\pi$?

Problem 3 (20=5+5+5+5 marks)
A time-dependent electric point dipole $\vec{d}(t)$, located at $\vec{r}=0$, has the charge density $\rho(\vec{r},t) = -\vec{d}(t)\cdot\vec{\nabla}\delta(\vec{r})$.

(a) Find the corresponding current density $\vec{j}(\vec{r},t)$.

(b) Verify that your choice for $\vec{j}(\vec{r},t)$ is such that the magnetic moment $\vec{\mu}(t) = \dfrac{1}{2c}\displaystyle\int (\mathrm{d}\vec{r})\,\vec{r}\times\vec{j}(\vec{r},t)$ vanishes.

(c) Verify that $\displaystyle\int (\mathrm{d}\vec{r})\,\vec{j}(\vec{r},t)$ has the correct value.

(d) Find the vector potential $\vec{A}(\vec{r},t)$ in the Lorentz gauge.

Problem 4 (20 marks)
An electric point dipole of constant strength $|\vec{d}(t)| = d$ is located at $\vec{r}=0$ and oriented in the x, y plane, and rotates around the z axis with constant angular velocity: $\vec{e}_z\cdot\vec{d}(t) = 0$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{d}(t) = \vec{\omega}\times\vec{d}(t)$ with $\vec{\omega} = \omega\vec{e}_z$. Find $\dfrac{\mathrm{d}P}{\mathrm{d}\Omega}$, the angular distribution of the radiated power, and the total radiated power $P = \displaystyle\int \mathrm{d}\Omega\,\frac{\mathrm{d}P}{\mathrm{d}\Omega}$, both averaged over one period of the rotation.