AY 2007/08, Sem 2, PC4242, 2nd Test

Question ..1/6......

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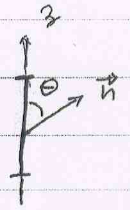
$\boxed{1}$ The expression for $\frac{dP}{d\Omega}(t)$ on page 76 of the notes applies, whereby here



$$\int (d\vec{r}\,')\, \frac{\partial}{\partial t}\, \vec{j}\,(\vec{r}\,', t_r)$$

$$= I\vec{e}_z \int_{-\pi c/\omega}^{\pi c/\omega} dz'\, \sin(\omega z'/c)\, \frac{\partial}{\partial t} \cos\left(\omega\left(t - \frac{r}{c} + \frac{z'}{c}\cos\theta\right)\right)$$

$$= -cI\vec{e}_z \int_{-\pi}^{\pi} d\phi\, \sin\phi\, \sin(\omega t_e + \phi\cos\theta)$$

with $t_e = t - \frac{r}{c}$ (emission time), and $\phi = \frac{\omega}{c} z'$.

Of

$$\sin(\omega t_e + \phi\cos\theta) = \sin(\omega t_e)\cos(\phi\cos\theta) + \cos(\omega t_e)\sin(\phi\cos\theta)$$

only the 2nd term contributes to the ϕ integral, giving

$$\int (d\vec{r}\,')\, \frac{\partial}{\partial t}\, \vec{j}\,(\vec{r}\,', t_r)$$

$$= \frac{1}{2} c I \vec{e}_z \cos(\omega t_e) \int_{-\pi}^{\pi} d\phi \left[\cos(\phi + \phi\cos\theta) - \cos(\phi - \phi\cos\theta)\right]$$

$$= cI\vec{e}_z \cos(\omega t_e)\left[\frac{\sin(\pi + \pi\cos\theta)}{1+\cos\theta} - \frac{\sin(\pi - \pi\cos\theta)}{1-\cos\theta}\right]$$

$$= -2cI\vec{e}_z \cos(\omega t_e)\, \frac{\sin(\pi\cos\theta)}{1-(\cos\theta)^2} \quad ,$$

Question 2/6

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and with $|\vec{n}\times\vec{e}_z|^2 = (\sin\theta)^2$, we obtain

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\,(2cI)^2\cos(\omega t_e)^2\left(\frac{\sin(\pi\cos\theta)}{\sin\theta}\right)^2 .$$

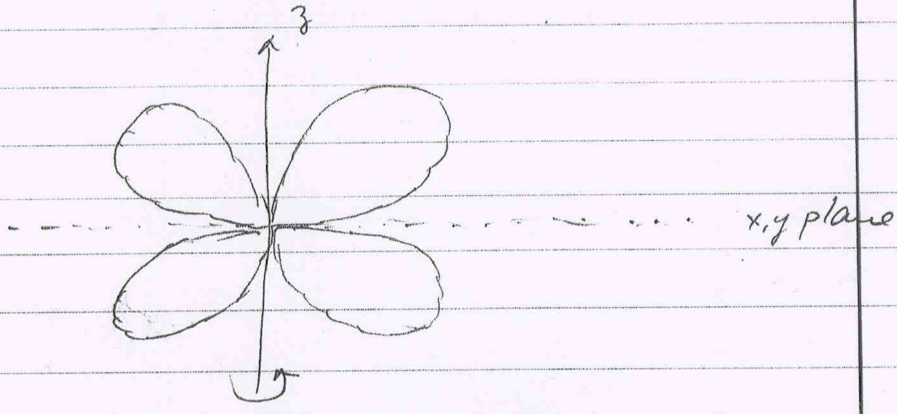
Upon averaging over one or more periods of the oscillation, $\cos(\omega t_e)^2 \to \frac{1}{2}$, this yields the result

$$\frac{dP}{d\Omega} = \frac{I^2}{2\pi c}\left(\frac{\sin(\pi\cos\theta)}{\sin\theta}\right)^2 .$$

For $\theta = 0, \frac{\pi}{2}, \pi$, the argument of $\sin(\pi\cos\theta)$ is $\pi, 0$, and $-\pi$, respectively, so that the numerator vanishes. For $\theta = 0, \pi$ we get an expression of the form $\frac{0}{0}$ which requires attention, but since the right-hand side does not change under the replacement $\theta \to \pi-\theta$, it is enough to consider the limit $\theta \to 0$. Applying l'Hospital's rule,

$$\lim_{\theta\to 0}\frac{\sin(\pi\cos\theta)}{\sin\theta} = \lim_{\theta\to 0}\frac{-\cos(\pi\cos\theta)\,\pi\sin\theta}{\cos\theta} = 0,$$

establishes $\frac{dP}{d\Omega} = 0$ for $\theta = 0, \frac{\pi}{2}, \pi$ and $\frac{dP}{d\Omega} \neq 0$ for all other θ values. We thus have the following rough picture for the angular distribution:



which is rotationally symmetric around the z axis along which the antenna is stretched out.

$\boxed{2}$ The expression for $\frac{dE(\omega)}{d\Omega}$ on page 104 applies, whereby here

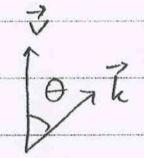
$$\vec{j}(\vec{r},t) = \begin{cases} 0 & \text{for } t<0\,, \\ -e\vec{v}\,\delta(\vec{r}-\vec{v}t) & \text{for } t>0\,, \end{cases}$$

so that

$$\vec{j}(\vec{k},\omega) = \int (d\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} \int_0^\infty dt\, e^{i\omega t} \left[-e\vec{v}\,\delta(\vec{r}-\vec{v}t)\right]$$

$$= -e\vec{v} \int_0^\infty dt\, e^{i\omega\left(1-\frac{v}{c}\cos\theta\right)t}$$

for $\vec{k}\cdot\vec{v}t = \frac{\omega}{c} v t \cos\theta$.



Question 4/6

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The integral is evaluated like the same one on page 107 of the notes, giving first

$$\vec{j}(\vec{k},\omega) = -e\vec{v}\,\frac{i}{\omega(1-\frac{v}{c}\cos\theta)}$$

and then

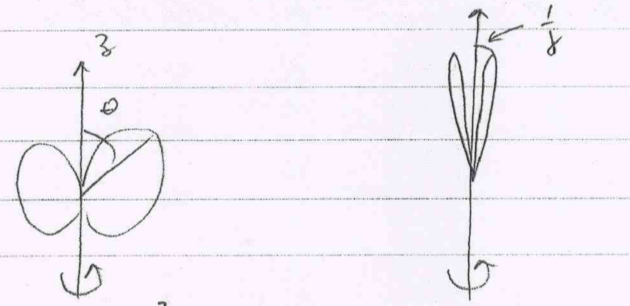
$$\vec{k}\times\vec{j}(\vec{k},\omega) = -e\,\vec{n}\times\frac{\vec{v}}{c}\,\frac{i}{1-\frac{v}{c}\cos\theta},$$

so that

$$\frac{dE(\omega)}{d\Omega} = \frac{1}{4\pi^2 c}\left|\vec{k}\times\vec{j}(\vec{k},\omega)\right|^2$$

$$= \frac{e^2}{4\pi^2 c^3}\left(\frac{\sin\theta}{1-\frac{v}{c}\cos\theta}\right)^2 .$$

There is no ω dependence, and this unphysical feature originates in the oversimplified description of the decay process as an instantaneous even of no duration. The resulting expression is identical with the one studied on pages 109 and 110 of the notes, so that we have the angular distribution found there:



$\propto (\sin\theta)^2$ for $\frac{v}{c} \ll 1$

strongly peaked in the near forward direction, with maximal intensity at $\theta = 1/\gamma$ for $\frac{v}{c} \lesssim 1$.

Question 5/6

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$\boxed{3}$ For $\beta \ll 1$, we need $J_m(z)$ for $0 \leq z \ll 1$ and $m>0$, for which the generating function tells us that $J_m(z) \propto z^m$:

$$J_m(z) = i^{-m} \int \frac{d\varphi}{2\pi} e^{-im\varphi} e^{iz\cos\varphi}$$

$$= i^{-m} \int \frac{d\varphi}{2\pi} e^{-im\varphi} \underbrace{\sum_{k=0}^{\infty} \frac{(iz)^k}{k!} \left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right)^k}_{\substack{\text{need } k\geq m \text{ for} \\ \text{nonzero contribution,} \\ \text{and get lowest power} \\ \text{for } k=m \text{ and} \\ (\cos\varphi)^m \to \left(\frac{e^{i\varphi}}{2}\right)^m,}}$$

giving $J_m(z) = \frac{1}{m!}\left(\frac{z}{2}\right)^m + O(z^{m+2})$ for the small-z form of $J_m(z)$. Accordingly, for $\beta \ll 1$ we have

$$\frac{dP_m}{d\Omega} \propto \beta^3 \beta^{2m-2} = \beta^{2m+1}$$

both for $\parallel$ and for $\perp$ polarization, and therefore only the $m=1$ term contributes in the limit $\beta \to 0$.

Thus, for $\beta \ll 1$

$$P_\parallel = \sum_{m=1}^{\infty} \int d\Omega \left(\frac{dP_m}{d\Omega}\right)_\parallel = \int d\Omega \left(\frac{dP_1}{d\Omega}\right)_\parallel$$

$$= \frac{\omega_0}{2\pi} \frac{e^2}{R} \beta^3 \underbrace{\int d\Omega \left(\frac{1}{2}\right)^2}_{\frac{1}{4}\cdot 4\pi} = \frac{1}{2} \omega_0 \frac{e^2}{R} \beta^3 ,$$

where $J_1'(\beta \sin\theta) = \frac{1}{2}$ enters, and

$$P_\perp = \int d\Omega \left(\frac{dP_1}{d\Omega}\right)_\perp = \frac{\omega_0}{2\pi} \frac{e^2}{R} \beta^3 \underbrace{\int d\Omega \left(\frac{1}{2}\cos\theta\right)^2}_{= \frac{1}{4} \cdot \frac{1}{3} \cdot 4\pi}$$

$$= \frac{1}{6}\, \omega_0 \frac{e^2}{R} \beta^3 .$$

Indeed, we have $\dfrac{P_\parallel}{P_\perp} = 3$ for $\beta \ll 1$.