

Write answers on this side of the paper only.

Do not write on
either margin

□ We have the electric dipole moment

$$\vec{d}(t) = \int (d\vec{r}) \vec{r} \rho(\vec{r}, t) \quad \text{with } (d\vec{r}) = ds \, dy \, dz$$

or

$$\vec{d}(t) = eR \int_{(2\pi)} \frac{dy}{2\pi} f(\varphi - \omega t) \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix}$$

so that

$$\begin{aligned} \ddot{\vec{d}} &= eR \omega^2 \int \frac{dy}{2\pi} f''(\varphi - \omega t) \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix} \\ &= -eR \omega^2 \int \frac{dy}{2\pi} f(\varphi - \omega t) \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix} \end{aligned}$$

after two integrations by part.

likewise, we have the magnetic dipole moment

$$\begin{aligned} \vec{\mu}(t) &= \frac{1}{2c} \int (d\vec{r}) \vec{r} \times \vec{j}(\vec{r}, t) \\ &= \frac{\omega R}{2c} \int (d\vec{r}) \rho(\vec{r}, t) \begin{pmatrix} -3 \cos \varphi \\ -3 \sin \varphi \\ 0 \end{pmatrix} \\ &= \frac{\omega R}{2c} eR \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int \frac{dy}{2\pi} f(\varphi - \omega t), \end{aligned}$$

which is constant in time, so that

$$\ddot{\vec{\mu}} = 0.$$

And for the electric quadrupole moment
we have

$$\vec{Q}(t) = \int (d\vec{r}) (3\vec{r}\vec{r} - r^2 \vec{1}) \rho(\vec{r}, t)$$

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or

$$\ddot{\mathbf{Q}}(t) = eR^2 \int_{2\pi} \frac{d\varphi}{2\pi} f(\varphi - \omega t) \begin{bmatrix} 3(\cos\varphi)^2 - 1 & 3\sin\varphi\cos\varphi & 0 \\ 3\sin\varphi\cos\varphi & 3(\sin\varphi)^2 - 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

so that, after three integrations by part,

$$\ddot{\mathbf{Q}}(t) = eR^2\omega^3 \int_{2\pi} \frac{d\varphi}{2\pi} f(\varphi - \omega t) 12 \begin{pmatrix} \sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

a) For $f(\varphi - \omega t) = \cos\varphi \cos(\omega t) + \sin\varphi \sin(\omega t)$,
we get

$$\ddot{\mathbf{d}}(t) = -eR\omega^2 \frac{1}{2} \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix} \quad \text{and} \quad \ddot{\mathbf{Q}}(t) = 0,$$

so that only $\ddot{\mathbf{d}}$ contributes to $\frac{dP}{d\Omega}$ as
given on page 82 in the notes. With

$$\hat{\mathbf{n}} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} \quad \text{we thus have}$$

$$\begin{aligned} |\hat{\mathbf{n}} \times \ddot{\mathbf{d}}|^2 &= \ddot{\mathbf{d}}^2 - (\hat{\mathbf{n}} \cdot \ddot{\mathbf{d}})^2 \\ &= \left(\frac{1}{2}eR\omega^2\right)^2 \left[1 - (\sin\theta \cos(\phi - \omega t))^2\right] \end{aligned}$$

and get

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left(\frac{1}{2}eR\omega^2\right)^2 \left[1 - \frac{1}{2}(\sin\theta)^2\right]$$

after the averaging over one period, which
amounts to $\cos(\phi - \omega t)^2 \rightarrow \frac{1}{2}$. And
the total radiated power is

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$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3c^3} \left(\frac{eR\omega^2}{2} \right)^2 = \frac{\omega}{6} \frac{e^2}{R} \left(\frac{\omega R}{c} \right)^3.$$

[6] For $f(\varphi - \omega t) = \cos(2\varphi) \cos(\omega t) + \sin(2\varphi) \sin(\omega t)$,
we have

$$\ddot{\vec{d}} = 0 \text{ and}$$

$$\ddot{\vec{Q}}(t) = 6e\omega^3 R^2 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

giving $\left| \vec{n} \times \ddot{\vec{Q}} \cdot \vec{n} \right|^2 = (6e\omega^3 R^2)^2 \left| \vec{n} \times \sin\theta \begin{pmatrix} \sin(\phi - 2\omega t) \\ \cos(\phi - 2\omega t) \\ 0 \end{pmatrix} \right|^2$
 $= (6e\omega^3 R^2)^2 \left[(\sin\theta)^2 - (\sin\theta)^4 \sin^2(2\phi - 2\omega t) \right]$

and then, after the time averaging,

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left(\frac{e\omega^3 R^2}{c} \right)^2 \left[(\sin\theta)^2 - \frac{1}{2} (\sin\theta)^4 \right],$$

and the total radiated power is

$$P = \frac{1}{c^3} \left(\frac{e\omega^3 R^2}{c} \right)^2 \left(\frac{2}{3} - \frac{4}{15} \right) = \frac{2}{5} \omega \frac{e^2}{R} \left(\frac{\omega R}{c} \right)^5.$$

[2] [a] With $t_r = t - \frac{r}{c} + \frac{1}{c} \vec{n} \cdot \vec{r}' = t_e + \frac{1}{c} (n_x x' + n_y y' + n_z z')$
we have (see page 76 in the notes)

$$\int (d\vec{r}') \frac{\partial}{\partial t} \vec{j}_{\pm}(\vec{r}', t_r)$$

$$= \vec{e}_z I \omega \int_{-L/2}^{L/2} dz' \sin\left(\omega t_e \pm \frac{\omega}{c} n_x \frac{a}{2} + \frac{\omega}{c} n_z z' \mp \frac{1}{2}\beta\right) \times \cos(\pi z'/L)$$

