[1] We have the electric dipole moment

$$\vec{d}(t) = \int (d\vec{r})\, \vec{r}\, \rho(\vec{r},t) \qquad \text{with } (d\vec{r}) = ds\, s\, d\varphi\, dz$$

or

$$\vec{d}(t) = eR \int_{(2\pi)} \frac{d\varphi}{2\pi} f(\varphi-\omega t) \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

so that

$$\ddot{\vec{d}} = eR\omega^2 \int \frac{d\varphi}{2\pi} f''(\varphi-\omega t) \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

$$= -eR\omega^2 \int \frac{d\varphi}{2\pi} f(\varphi-\omega t) \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

after two integrations by part.

Likewise, we have the magnetic dipole moment

$$\vec{\mu}(t) = \frac{1}{2c} \int (d\vec{r})\, \vec{r} \times \vec{j}(\vec{r},t)$$

$$= \frac{\omega R}{2c} \int (d\vec{r})\, \rho(\vec{r},t) \begin{pmatrix} -z\cos\varphi \\ -z\sin\varphi \\ s \end{pmatrix}$$

$$= \frac{\omega R}{2c} eR \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int_{(2\pi)} \frac{d\varphi}{2\pi} f(\varphi-\omega t),$$

which is constant in time, so that

$$\ddot{\vec{\mu}} = 0.$$

And for the electric quadrupole moment we have

$$\overleftrightarrow{Q}(t) = \int (d\vec{r}) \left(3\vec{r}\vec{r} - r^2 \overleftrightarrow{1}\right) \rho(\vec{r},t)$$

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or

$$\overleftrightarrow{Q}(t) = eR^2 \int_{(2\pi)} \frac{d\varphi}{2\pi} f(\varphi-\omega t)\left[\begin{pmatrix} 3(\cos\varphi)^2-1 & 3\sin\varphi\cos\varphi & 0 \\ 3\sin\varphi\cos\varphi & 3(\sin\varphi)^2-1 & 0 \\ 0 & 0 & -1\end{pmatrix}\right]$$

so that, after three integrations by part,

$$\dddot{\overleftrightarrow{Q}}(t) = eR^2\omega^3 \int_{(2\pi)} \frac{d\varphi}{2\pi} f(\varphi-\omega t)\; 12 \begin{pmatrix} \sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0\end{pmatrix}.$$

$\boxed{a}$ For $f(\varphi-\omega t) = \cos\varphi\cos(\omega t) + \sin\varphi\sin(\omega t)$, we get

$$\ddot{\vec{d}}(t) = -eR\omega^2\,\frac{1}{2}\begin{pmatrix}\cos(\omega t)\\ \sin(\omega t)\\ 0\end{pmatrix} \quad \text{and} \quad \dddot{\overleftrightarrow{Q}}(t) = 0,$$

so that only $\ddot{\vec{d}}$ contributes to $\frac{dP}{d\Omega}$ as given on page 82 in the notes. With $\vec{n} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$ we thus have

$$\left|\vec{n}\times\ddot{\vec{d}}\right|^2 = \ddot{\vec{d}}^{\,2} - (\vec{n}\cdot\ddot{\vec{d}})^2$$
$$= \left(\tfrac{1}{2}eR\omega^2\right)^2\left[1-\left(\sin\theta\cos(\phi-\omega t)\right)^2\right]$$

and get

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left(\tfrac{1}{2}eR\omega^2\right)^2\left[1-\tfrac{1}{2}(\sin\theta)^2\right]$$

after the averaging over one period, which amounts to $\cos(\phi-\omega t)^2 \to \frac{1}{2}$. And the total radiated power is

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$$P = \int d\Omega \, \frac{dP}{d\Omega} = \frac{2}{3c^3}\left(\frac{eR\omega^2}{2}\right)^2 = \frac{\omega}{6}\,\frac{e^2}{R}\left(\frac{\omega R}{c}\right)^3 .$$

$\boxed{b}$ For $f(\varphi-\omega t) = \cos(2\varphi)\cos(\omega t) + \sin(2\varphi)\sin(\omega t)$, we have

$$\ddot{\vec{d}} = 0 \quad \text{and}$$

$$\dddot{\overleftrightarrow{Q}}(t) = 6e\omega^3R^2 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

giving

$$\left|\vec{n}\times \dddot{\overleftrightarrow{Q}}\cdot\vec{n}\right|^2 = \left(6e\omega^3R^2\right)^2 \left|\vec{n}\times \sin\theta \begin{pmatrix} \sin(\phi-2\omega t) \\ \cos(\phi-2\omega t) \\ 0 \end{pmatrix}\right|^2$$

$$= \left(6e\omega^3R^2\right)^2\left[(\sin\theta)^2 - (\sin\theta)^4 \sin(2\phi-2\omega t)^2\right]$$

and then, after the time averaging,

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left(\frac{e\omega^3R^2}{c}\right)^2\left[(\sin\theta)^2 - \frac{1}{2}(\sin\theta)^4\right],$$

and the total radiated power is

$$P = \frac{1}{c^3}\left(\frac{e\omega^3R^2}{c}\right)^2\left(\frac{2}{3}-\frac{4}{15}\right) = \frac{2}{5}\,\omega\,\frac{e^2}{R}\left(\frac{\omega R}{c}\right)^5 .$$

$\boxed{2}\boxed{a}$ With $t_r = t - \frac{r}{c} + \frac{1}{c}\vec{n}\cdot\vec{r}\,' = t_e + \frac{1}{c}(n_x x' + n_y y' + n_z z')$ we have (see page 76 in the notes)

$$\int (d\vec{r}\,')\,\frac{\partial}{\partial t}\,\vec{j}_\pm(\vec{r}\,', t_r)$$

$$= -\vec{e}_z\, I\omega \int_{-L/2}^{L/2} dz' \sin\left(\omega t_e \pm \frac{\omega}{c} n_x \frac{a}{2} + \frac{\omega}{c} n_z z' \mp \frac{1}{2}\beta\right) \times \cos(\pi z'/L)$$

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where $\sin(\dots) \to \sin\left(\omega t_e \pm \left(\frac{\omega a}{2c} n_x - \frac{\beta}{2}\right)\right) \cos\left(\frac{\omega}{c} n_z z'\right)$ in view of the even factor $\cos\left(\pi \frac{z'}{L}\right)$, so that

$$\int (d\vec{r}\,') \frac{\partial}{\partial t} \vec{j}_\pm(\vec{r}\,'; t_r)$$

$$= -\vec{e}_z \,\omega I \sin\left(\omega t_e \pm \left(\frac{\omega a}{2c} n_x - \frac{\beta}{2}\right)\right)$$

$$\times \int_{-L/2}^{L/2} dz' \cos\left(\frac{\omega}{c} n_z z'\right) \cos\left(\frac{\pi}{L} z'\right)$$

$$= -\vec{e}_z \,\omega I \sin\left(\omega t_e \pm \left(\frac{\omega a}{2c} n_x - \frac{\beta}{2}\right)\right) \frac{2L}{\pi} \frac{\cos\left(\frac{\pi}{2} n_z\right)}{1 - n_z^2}$$

with $n_x = \sin\theta \cos\phi$, $n_z = \cos\theta$, $1 - n_z^2 = (\sin\theta)^2$.

Accordingly,

$$\vec{n} \times \int (d\vec{r}\,') \frac{\partial}{\partial t} \vec{j}(\vec{r}\,'; t_r) = \vec{n} \times \int (d\vec{r}\,') \frac{\partial}{\partial t} (\vec{j}_+ + \vec{j}_-)$$

$$= -\vec{n} \times \vec{e}_z \, 2cI \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{(\sin\theta)^2} \, 2\sin(\omega t_e) \cos\left(\frac{\omega a}{2c} n_x - \frac{\beta}{2}\right),$$

which gives, after the time averaging

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} (2cI)^2 \left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2 2\cos\left(\frac{\omega a}{2c} n_x - \frac{\beta}{2}\right)^2$$

$$= \underbrace{\frac{2I^2}{\pi c} \left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2}_{4\times \text{ single antenna pattern}} \underbrace{\cos\left(\frac{\pi a}{2L} \sin\theta\cos\phi - \frac{\beta}{2}\right)^2}_{\text{interference pattern.}}.$$

$\boxed{b}$ For emission into the directions $\vec{n} = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}$, that is: $\sin\theta = 1$, $\cos\theta = 0$, $\cos\phi = \pm 1$, we have

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$$\frac{dP}{d\Omega}(\pm\vec{e}_x) = \frac{2I^2}{\pi c}\cos\left(\frac{\pi a}{2L} \mp \frac{\beta}{2}\right)^2 \text{ and we get}$$

$$\frac{dP}{d\Omega}(+\vec{e}_x) = \frac{2I^2}{\pi c} \quad (\text{maximal}) \quad \text{and}$$

$$\frac{dP}{d\Omega}(-\vec{e}_x) = 0 \quad (\text{minimal})$$

if $\frac{\pi a}{2L} = \frac{\beta}{2} = \frac{\pi}{4}$ or $\underline{\underline{a = \frac{1}{2}L \text{ and } \beta = \frac{1}{2}\pi}}$.

$\boxed{3}$ In the context of the Exercises 16 and 21, it was noted that

$$\frac{\partial}{\partial t} t_{ret} = \left[1 - \frac{1}{c}\vec{n}\cdot\dot{\vec{R}}(t_{ret})\right]^{-1}$$

with $\vec{n} = \dfrac{\vec{r} - \vec{R}(t_{ret})}{|\vec{r} - \vec{R}(t_{ret})|}$, and

$$\vec{\nabla} t_{ret} = -\frac{\vec{n}}{c}\left[1 - \frac{1}{c}\vec{n}\cdot\dot{\vec{R}}(t_{ret})\right]^{-1}.$$

From these statements, it follows immediately that

$$\left(\vec{\nabla} t_{ret}\right)^2 - \left(\frac{1}{c}\frac{\partial}{\partial t} t_{ret}\right)^2 = 0.$$

$\boxed{4}$ In $-\left.\frac{dE}{dt}\right|_{rad} = \frac{2e^2}{3c^3}\left(1 - \left(\frac{v}{c}\right)^2\right)^{-3}\left[\left(\frac{d\vec{v}}{dt}\right)^2 - \left(\frac{\vec{v}}{c}\times\frac{d\vec{v}}{dt}\right)^2\right]^2$

(see Exercise 43), we insert the expression

$$\frac{d}{dt}\vec{v} = \vec{\omega}_0\times\vec{v}, \qquad v = \omega_0 R,$$

$\vec{v}\times\frac{d}{dt}\vec{v} = \vec{\omega}_0 v^2$, that apply to motion

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in a circle with constant speed, and get

$$-\frac{dE}{dt}\Big|_{rad} = \frac{2e^2}{3c^3}\left(1-\left(\frac{v}{c}\right)^2\right)^{-3}\underbrace{\left[\omega_0^2 v^2 - \left(\frac{1}{c}\omega_0 v^2\right)^2\right]}_{=(\omega_0 v)^2\left(1-\left(\frac{v}{c}\right)^2\right)}$$

$$= \frac{2e^2}{3c^3}\left(1-\left(\frac{v}{c}\right)^2\right)^{-2}(\omega v)^2$$

$$= \frac{2}{3}\omega_0 \frac{e^2}{R}\left(\frac{v}{c}\right)^3\left(1-\left(\frac{v}{c}\right)^2\right)^{-2},$$

which is the total radiated power of synchrotron radiation (page 132 of the notes), indeed.