Do not write on either margin

$\boxed{1}$ We use the infinitesimal version of the 4-dyadic of (3.1.10),

$$L(\delta\vec{v}) = \begin{pmatrix} 1 & \delta\vec{v}^T/c \\ \delta\vec{v}/c & \overleftrightarrow{1} \end{pmatrix},$$

and first get

$$L(\delta\vec{v}_2)\, L(\delta\vec{v}_1) = \begin{pmatrix} 1+\delta\vec{v}_2\cdot\delta\vec{v}_1/c^2 & (\delta\vec{v}_2+\delta\vec{v}_1)^T/c \\ (\delta\vec{v}_2+\delta\vec{v}_1)/c & \overleftrightarrow{1}+\delta\vec{v}_2\,\delta\vec{v}_1^T/c^2 \end{pmatrix}$$

as well as

$$L(-\delta\vec{v}_2)\, L(-\delta\vec{v}_1) = \begin{pmatrix} 1+\delta\vec{v}_2\cdot\delta\vec{v}_1/c^2 & -(\delta\vec{v}_2+\delta\vec{v}_1)^T/c \\ -(\delta\vec{v}_2+\delta\vec{v}_1)/c & \overleftrightarrow{1}+\delta\vec{v}_2\,\delta\vec{v}_1^T/c^2 \end{pmatrix}.$$

and thus establish

$$L(-\delta\vec{v}_2)\, L(-\delta\vec{v}_1)\, L(\delta\vec{v}_2)\, L(\delta\vec{v}_1) = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & \overleftrightarrow{1}+(\delta\vec{v}_2\,\delta\vec{v}_1^T-\delta\vec{v}_1\,\delta\vec{v}_2^T)/c^2 \end{pmatrix},$$

so that the effect on t and $\vec{r}$ is

$$\begin{pmatrix} ct \\ \vec{r} \end{pmatrix} \longrightarrow \begin{pmatrix} ct \\ \vec{r}+(\delta\vec{v}_2\,\delta\vec{v}_1\cdot\vec{r}-\delta\vec{v}_1\,\delta\vec{v}_2\cdot\vec{r})/c^2 \end{pmatrix}$$

or $\delta t = 0$ and

$$\delta\vec{r} = \frac{1}{c^2}(\delta\vec{v}_1\times\delta\vec{v}_2)\times\vec{r} = \overrightarrow{\delta\phi}\times\vec{r},$$

which is a rotation by the infinitesimal vectorial angle $\overrightarrow{\delta\phi} = \frac{1}{c^2}(\delta\vec{v}_1\times\delta\vec{v}_2)$.

$\boxed{2}$ We have

$$\partial_\nu T^{\mu\nu} = \partial_\nu \left(\frac{1}{4\pi} F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{16\pi} g^{\mu\nu} F^{\kappa\lambda} F_{\kappa\lambda} \right)$$

$$= \frac{1}{4\pi} F^{\mu\lambda} \partial_\nu F^{\nu}{}_{\lambda} + \frac{1}{4\pi} F_{\nu\lambda} \partial^\nu F^{\mu\lambda}$$

$$- \frac{1}{16\pi} \partial^\mu \left(F^{\kappa\lambda} F_{\kappa\lambda} \right)$$

where $\partial_\nu F^{\nu}{}_{\lambda} = -\frac{4\pi}{c} j_\lambda$ (see (4.2.13)) and, as a consequence of (4.2.16),

$$F_{\nu\lambda} \partial^\nu F^{\mu\lambda} = \frac{1}{2} F_{\nu\lambda} \partial^\nu F^{\mu\lambda} - \frac{1}{2} F_{\nu\lambda} \partial^\lambda F^{\nu\mu} - \frac{1}{2} F_{\nu\lambda} \partial^\mu F^{\lambda\nu}$$

$$= \frac{1}{2} \underbrace{\left(F_{\nu\lambda} \partial^\nu F^{\mu\lambda} - F_{\lambda\nu} \partial^\lambda F^{\mu\nu} \right)}_{=0} - \frac{1}{4} \partial^\mu \left(F_{\nu\lambda} F^{\lambda\nu} \right)$$

$$= \frac{1}{4} \partial^\mu \left(F^{\lambda\nu} F_{\lambda\nu} \right),$$

so that

$$\partial_\nu T^{\mu\nu} = - F^{\mu\lambda} \frac{1}{c} j_\lambda + \frac{1}{16\pi} \partial^\mu \underbrace{\left(F^{\lambda\nu} F_{\lambda\nu} - F^{\kappa\lambda} F_{\kappa\lambda} \right)}_{=0}$$

$$= - F^{\mu\nu} \frac{1}{c} j_\nu ,$$

indeed. Using ordinary matrix multiplication, we evaluate

$$F^{\mu\nu} \frac{1}{c} j_\nu = F^{\mu}{}_{\nu} \frac{1}{c} j^\nu$$

$$= \begin{pmatrix} 0 & \vec{E}^T \\ \vec{E} & -\vec{B} \times \overleftrightarrow{1} \end{pmatrix} \begin{pmatrix} \rho \\ \frac{1}{c} \vec{j} \end{pmatrix} = \begin{pmatrix} \frac{1}{c} \vec{j} \cdot \vec{E} \\ \rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \end{pmatrix},$$

which are the components of the contravariant 4-vector field of the force density, see (3.3.24). Together, then,

$$\partial_\nu T^{\mu\nu} = -f^\mu ,$$

which is the contravariant version of the covariant equation in (3.4.14).

$\boxed{3}$ All points on the circle at $z=0$ are the distance $\sqrt{R^2+z^2}$ away from the point $\vec{r} = z\vec{e}_z$ on the z axis. Therefore, the retardation condition (5.3.3) reads

$$t_{ret} + \frac{1}{c}\sqrt{R^2+z^2} = t ,$$

and since $\vec{R}(t_{ret}) \perp \vec{V}(t_{ret})$ as well as $\vec{r} \perp \vec{V}(t_{ret})$ in (5.3.6), for <u>this</u> geometry, we have

$$\Phi(\vec{r},t) = \frac{e}{\sqrt{R^2+z^2}} \quad \text{for } \vec{r} = z\vec{e}_z$$

and

$$\vec{A}(\vec{r},t) = \frac{1}{c}\vec{V}(t_{ret})\frac{e}{\sqrt{R^2+z^2}} \quad \text{for } \vec{r} = z\vec{e}_z$$

and $\vec{V}(t_{ret}) = v\left(-\vec{e}_x \sin\varphi + \vec{e}_y \cos\varphi\right)$

with $\varphi = v t_{ret}/R = vt/R - \frac{v}{c}\sqrt{1+(z/R)^2}$.

We can only infer $E_z(\vec{r},t)$ since all other components of $\vec{E}$ and $\vec{B}$ require some knowlege of the x,y dependence of the potentials.

$\boxed{4}$ The continuity equation tells us that

$$\frac{\partial}{\partial t}\rho(\vec{r},t) = -I\,\delta(x)\delta(y)\cos(\omega t)\frac{\partial}{\partial z}\left[\eta(L^2-4z^2)\cos(\pi z/L)\right]$$

$$= I\,\delta(x)\delta(y)\cos(\omega t)\,\eta(L^2-4z^2)\,\frac{\pi}{L}\sin(\pi z/L)$$

so that

$$\rho(\vec{r},t) = \frac{\pi I}{\omega L}\,\delta(x)\delta(y)\,\eta(L^2-4z^2)\sin\left(\frac{\pi z}{L}\right)\sin(\omega t).$$

We thus have the electric dipole moment

$$\vec{d}(t) = \int(d\vec{r})\,\vec{r}\,\rho(\vec{r},t) = \frac{\pi I}{\omega L}\sin(\omega t)\underbrace{\int_{-L/2}^{L/2}dz\,z\sin\left(\frac{\pi z}{L}\right)}_{=\frac{2L^2}{\pi^2}}$$

$$= \frac{2IL}{\pi\omega}\sin(\omega t)$$

and its second time derivative

$$\ddot{\vec{d}}(t) = -\frac{2}{\pi}IL\omega\sin(\omega t),$$

whereas the magnetic dipole moment,

$$\vec{\mu}(t) = \int(d\vec{r})\,\frac{1}{2c}\,\vec{r}\times\vec{j}(\vec{r},t)$$

$$= \frac{1}{2c}I\cos(\omega t)\int_{-L/2}^{L/2}dz\;z\,\vec{e}_z\times\vec{e}_z\cos\frac{\pi z}{L} = 0,$$

and the electric quadrupole moment,

$$\overleftrightarrow{Q}(t) = \int(d\vec{r})\,\underbrace{\left(3\vec{r}\vec{r}-r^2\overleftrightarrow{1}\right)}_{\text{even in }\vec{r}}\,\underbrace{\rho(\vec{r},t)}_{\text{odd in }\vec{r}} = 0,$$

Question 5/5

Write answers on this side of the paper only.

Do not write on either margin

vanish both. Accordingly, (6.3.16) reduces to (6.3.4) and (6.3.6) applies,

$$P = \frac{2}{3c^3}\left(\frac{2}{\pi} I L \omega\right)^2 \sin(\omega t_e)^2$$

$$\xrightarrow[\text{average}]{\text{time}} \frac{1}{3c^3}\left(\frac{2}{\pi} I L \omega\right)^2 \underset{\omega = 2\pi c/\lambda}{=} \frac{16}{3}\frac{I^2}{c}\left(\frac{L}{\lambda}\right)^2.$$