PC 4242, AY 2008/09, Sem 2, Test 2

Question 1/4

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{1}$ $$P(T) = \int d\Omega \, \frac{dP(T)}{d\Omega}$$

$$= \frac{\omega_0}{16\pi}\frac{e^2}{R}\frac{v}{c}\,2\pi\int_0^\pi d\vartheta\,\sin\vartheta\left[\frac{1+3\left(\frac{v}{c}\right)^2}{\left(1-\frac{v}{c}\cos\vartheta\right)^3} - \frac{1-\left(\frac{v}{c}\right)^2}{\left(1-\frac{v}{c}\cos\vartheta\right)^4}\right]$$

(9.8.4)

$$= \frac{\omega_0}{8}\frac{e^2}{R}\frac{v}{c}\left[-\frac{1}{2\frac{v}{c}}\frac{1+3\left(\frac{v}{c}\right)^2}{\left(1-\frac{v}{c}\cos\vartheta\right)^2} + \frac{1}{3\frac{v}{c}}\frac{1-\left(\frac{v}{c}\right)^2}{\left(1-\frac{v}{c}\cos\vartheta\right)^3}\right]_{\vartheta=0}^{\vartheta=\pi}$$

$$= \frac{\omega_0}{8}\frac{e^2}{R}\frac{v}{c}\frac{1}{\left(1-\left(\frac{v}{c}\right)^2\right)^2}\left[2\left(1+3\left(\frac{v}{c}\right)^2\right) - \left(2+\frac{2}{3}\left(\frac{v}{c}\right)^2\right)\right]$$

$$= \frac{2}{3}\,\omega_0\,\frac{e^2}{R}\left(\frac{v}{c}\right)^3\frac{1}{\left(1-\left(\frac{v}{c}\right)^2\right)^2}$$

$$= \frac{2}{3}\,\omega_0\,\frac{e^2}{R}\left(\frac{v}{c}\right)^3\left(\frac{E}{mc^2}\right)^4,$$ indeed.

$\boxed{2}$ We have $\vec{d}(t) = \frac{\epsilon-1}{\epsilon+2}R^3\vec{E}(t)$ and $\frac{d^2}{dt^2}\vec{E}(t) = -\omega^2\vec{E}(t)$, so that Larmor's formula (6.3.6) gives

$$P = \frac{2}{3c^3}\,\omega^4\left(\frac{\epsilon-1}{\epsilon+2}R^3\right)^2\underbrace{\left|\vec{E}(t)\right|^2}_{=\frac{4\pi}{c}|\vec{S}|} = \sigma|\vec{S}|$$

with

$$\sigma = \frac{8\pi}{3}\left(\frac{\omega}{c}\right)^4\left(\frac{\epsilon-1}{\epsilon+2}R^3\right)^2.$$

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$\boxed{3}$ (a) We apply (11.2.2),

$$E(x,z) \cong -\frac{E_0}{2\pi}\frac{\partial}{\partial z}\int_0^\infty dx'\int_{-\infty}^\infty dy' \frac{e^{ik\sqrt{(x-x')^2+y'^2+z^2}}}{\sqrt{(x-x')^2+y'^2+z^2}}$$

and recognize that the y' integral is the same as in (11.6.23), so that

$$E(x,z) \cong -\frac{E_0}{2\pi}\frac{\partial}{\partial z}\int_0^\infty dx'\sqrt{\frac{2\pi i}{k\sqrt{(x-x')^2+z^2}}}\, e^{ik\sqrt{(x-x')^2+z^2}}$$

$$\cong -\frac{E_0}{2\pi}\sqrt{\frac{2\pi i}{kz}}\, ik\, e^{ikz}\int_0^\infty dx'\, e^{i\frac{1}{2}k\frac{(x-x')^2}{z}}$$

after employing the usual approximations. Substitute $x' = x + \sqrt{\pi z/k}\, t$ to arrive at

$$\frac{E(x,z)}{E_0} \cong \frac{e^{ikz}}{\sqrt{2i}}\int_{-T}^\infty dt\, e^{i\frac{\pi}{2}t^2} = \frac{e^{ikz}}{\sqrt{2i}}\int_{-\infty}^T dt\, e^{i\frac{\pi}{2}t^2}$$

with $T = \sqrt{\frac{k}{\pi z}}\, x = \sqrt{\frac{2}{\lambda z}}\, x$, so that

$$\left|\frac{E(x,z)}{E_0}\right|^2 = \frac{1}{2}\left|\int_{-\infty}^T dt\, e^{i\frac{\pi}{2}t^2}\right|^2$$

$$= \frac{1}{2}\left|\frac{1+i}{2} + \underbrace{\int_0^T dt\, e^{i\frac{\pi}{2}t^2}}_{F(T)}\right|^2$$

or

$$\left|\frac{E(x,z)}{E_0}\right|^2 = \frac{1}{4}\left|1 + (1-i)\,F(T)\right|^2.$$

(b) Since $F(T) = -F(-T)$, we have

$$F(T) = \mathrm{sgn}(T)\, F(|T|)$$

$$= \mathrm{sgn}(T)\left[\int_0^\infty dt\, e^{i\frac{\pi}{2}t^2} - \int_{|T|}^\infty dt\, e^{i\frac{\pi}{2}t^2}\right]$$

$$= \mathrm{sgn}(T)\left[\frac{1+i}{2} - \int_{|T|}^\infty dt\, \frac{1}{i\pi t}\frac{d}{dt} e^{i\frac{\pi}{2}t^2}\right]$$

$$= \mathrm{sgn}(T)\left[\frac{1+i}{2} + \frac{1}{i\pi|T|} e^{i\frac{\pi}{2}T^2} - \underbrace{\int_{|T|}^\infty dt\, \frac{e^{i\frac{\pi}{2}t^2}}{i\pi t^2}}_{O((\frac{1}{T})^3)}\right].$$

$$\cong \mathrm{sgn}(T)\frac{1+i}{2} + \frac{1}{i\pi T} e^{i\frac{\pi}{2}T^2},$$

valid for $|T| \gg 1$. For $x \gg \sqrt{\lambda z}$, $T \gg 1$, then, we have

$$\left|\frac{E(x,z)}{E_0}\right|^2 = \frac{1}{4}\left|1 + 1 + \underbrace{\frac{1-i}{i\pi T}}_{= \frac{\sqrt{2}}{\pi T} e^{-i\frac{3\pi}{4}}} e^{i\frac{\pi}{2}T^2}\right|$$

$$= 1 + \frac{\sqrt{2}}{\pi T}\cos\left(\frac{\pi}{2}T^2 - \frac{3\pi}{4}\right)$$

$$= 1 + \frac{\sqrt{\lambda z}}{\pi x}\cos\left(\frac{\pi x^2}{\lambda z} - \frac{3\pi}{4}\right)$$

after consistently discarding terms of relative size $(\frac{1}{T})^2$ or smaller.

And for $-x \gg \sqrt{\lambda z}$, $-T \gg 1$, we get

$$\left|\frac{E(x,z)}{E_0}\right|^2 = \frac{1}{4}\left|1-1+\frac{\sqrt{2}}{\pi T}e^{i\left(\frac{\pi}{2}T^2-\frac{3\pi}{4}\right)}\right|^2$$

$$= \frac{1}{(\sqrt{2}\pi T)^2} = \frac{\lambda z}{(2\pi x)^2} .$$

The first maximum should occur near $x=\sqrt{\frac{3}{4}\lambda z}$, where the argument of the cosine equals 0, and we estimate

$$1+\frac{\sqrt{\lambda z}}{\pi\sqrt{\frac{3}{4}\lambda z}} = 1+\frac{2}{\pi\sqrt{3}} \cong 1+\frac{2}{5} = 1.4$$

for the height of the first maximum. This agrees well with what the figure tells.