AY 2008/09, Sem 2, PC4242, Exam

Question.....1/5.....

Write answers on this side of the paper only.

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$\boxed{1}$ (a) According to (3.5.7) we have here $\vec{A}(\vec{r},t) = \frac{\vec{v}}{c}\Phi(\vec{r},t)$, so that

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\vec{v}}{c}\times\left(-\vec{\nabla}\Phi\right) = \frac{\vec{v}}{c}\times\left(-\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\vec{A}}_{\propto \vec{v}/c}-\vec{\nabla}\Phi\right)$$

$$= \frac{\vec{v}}{c}\times\vec{E}.$$

(b) Lorentz force $\vec{F} = e\,\vec{E}(\vec{v}t+a,t) + e\frac{\vec{v}}{c}\times\vec{B}(\vec{v}t+\vec{a},t)$

$$= e\left[\vec{E} + \frac{\vec{v}}{c}\times\left(\frac{\vec{v}}{c}\times\vec{E}\right)\right]$$

$$= e\left(1-\left(\frac{v}{c}\right)^2\right)\vec{E} + e\,\frac{\vec{v}}{c}\,\frac{\vec{v}}{c}\cdot\vec{E},$$

and from Exercise 13 we recall that

$$\vec{E}(\vec{r},t) = \frac{\gamma e}{R^3}\left(\vec{r}-\vec{v}t\right)$$

with $R = \sqrt{x^2+y^2+\gamma^2(z-vt)^2}$, so that

$$\vec{E}(\vec{v}t+\vec{a},t) = \frac{\gamma e}{R^3}\,\vec{a} \quad \text{with } R = \sqrt{\vec{a}_\perp^{\,2}+\gamma^2\vec{a}_\parallel^{\,2}}.$$

Together, these say that $\left(\frac{\vec{v}}{c}\frac{\vec{v}}{c}\cdot\vec{a} = \left(\frac{v}{c}\right)^2\vec{a}_\parallel = \frac{\gamma^2-1}{\gamma^2}\vec{a}_\parallel\right)$

$$\vec{F} = \frac{e^2}{\gamma R^3}\vec{a}_\perp + \frac{\gamma e^2}{R^3}\vec{a}_\parallel.$$

(c) If $\vec{a}=\vec{a}_\perp$, $\vec{a}_\parallel=0$, then $R=|\vec{a}_\perp|$ and $\vec{F} = \frac{e^2\vec{a}}{\gamma a^3}$.
There is no Lorentz contraction perpendicular to $\vec{v}$, so that a is also the distance in the rest frame. It follows that the force is smaller by a factor $\frac{1}{\gamma}$ than the force in the rest frame.
By contrast, if $\vec{a}=\vec{a}_\parallel$, $\vec{a}_\perp=0$, then $R=\gamma|\vec{a}_\parallel|=\gamma a$

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and $\vec{F} = e^2 \frac{(\gamma\vec{a})}{(\gamma a)^3}$ where $\gamma\vec{a}$ is the distance vector in the rest frame and γa is its length, so that the force is the same as in the rest frame.

$\boxed{2}$ Since the charged particle moves parallel to the surface of the dielectric, total internal reflection prevents the Čerenkov radiation from leaving the dielectric. Answer: The fraction is <u>zero</u>.

$\boxed{3}$ We apply the Huygens approximation and get, as a simple modification of (11.3.3)

$$E_x(\vec{r}) \cong -E_0 \frac{e^{ikr}}{r} \frac{i}{k\theta^2} \int_{ka\theta}^{kb\theta} dt\, t\, J_0(t)$$

$$= -E_0 \frac{e^{ikr}}{r} \frac{i}{k\theta^2} \left[kb\theta\, J_1(kb\theta) - ka\theta\, J_1(ka\theta)\right]$$

and, see (11.3.8),

$$\frac{d\sigma}{d\Omega} = r^2 \left|\frac{E_x(\vec{r})}{E_0}\right|^2 \cong \left|\frac{b}{\theta} J_1(kb\theta) - \frac{a}{\theta} J_1(ka\theta)\right|^2,$$

valid for $\theta \ll 1$, which are the relevant θ values.

$\boxed{4}$ Equation of motion is $\frac{d}{dt}\vec{p} = e\frac{\vec{v}}{c}\times\vec{B}$ with $\vec{p} = m\gamma\vec{v}$, or

$$\frac{d}{dt}\vec{v} = 0 \quad \text{for} \quad z<0 \text{ and } z>L$$

and

$$\frac{d}{dt}\vec{v} = \frac{eB}{\gamma mc}\begin{pmatrix} -v_z \sin(k_0 z) \\ v_z \cos(k_0 z) \\ v_x \sin(k_0 z) - v_y \cos(k_0 z) \end{pmatrix} \quad \text{for } 0<z<L.$$

In particular, $\frac{d}{dt} v_x = \frac{eB}{\gamma mc} \frac{1}{k_0} \frac{d}{dt} \cos(k_0 z)$ and

$$\frac{d}{dt} v_y = \frac{eB}{\gamma mc} \frac{1}{k_0} \frac{d}{dt} \sin(k_0 z) ,$$

so that

$$v_x = v_\perp \cos(k_0 z) , \quad v_y = v_\perp \sin(k_0 z)$$

are solutions provided that

$$\gamma v_\perp = \frac{eB}{mc k_0} .$$

Then

$$\frac{d}{dt} v_z = \frac{eB}{\gamma mc} \left[v_\perp \cos(k_0 z) \sin(k_0 z) - v_\perp \sin(k_0 z) \cos(k_0 z) \right]$$

$$= 0 \quad \text{or} \quad v_z(t) = v_\parallel$$

and $\quad z(t) = v_\parallel t$, as required by consistency.

(b) Here $\vec{j}(\vec{r},t) = e\,\vec{v}(t)\,\delta(\vec{r} - \vec{r}(t))$

↖ trajectory of the electron, $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$

so that

$$\vec{j}(\vec{k},t) = \int (d\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, \vec{j}(\vec{r},t) = e\,\vec{v}(t)\, e^{-i\vec{k}\cdot\vec{r}(t)}$$

and for $\vec{k} = \frac{\omega}{c}\vec{n} = \frac{\omega}{c}\vec{e}_z$ we get

$$\frac{dE(\omega)}{d\Omega} = \frac{\omega^2 e^2}{4\pi^2 c^3} \left| \int_0^T dt\, e^{i\omega t} e^{-i\frac{\omega}{c} v_\parallel t}\, \vec{e}_z \times \vec{v}(t) \right|^2$$

with $\vec{e}_z \times \vec{v}(t) = v_x \vec{e}_y - v_y \vec{e}_x$

$$= \frac{1}{2} v_\perp \left[(\vec{e}_y + i\vec{e}_x)\, e^{ik_0 v_\parallel t} + (\vec{e}_y - i\vec{e}_x)\, e^{-ik_0 v_\parallel t} \right].$$

The t integrals are of the form

$$\int_0^T dt\, e^{i\Omega t} = e^{i\frac{\Omega T}{2}} \frac{\sin\frac{\Omega T}{2}}{\frac{\Omega}{2}}$$

for $\Omega = \left(1-\frac{v_\parallel}{c}\right)\omega + k_0 v_\parallel$ and $\Omega = \left(1-\frac{v_\parallel}{c}\right)\omega - k_0 v_\parallel$,
where $k_0 v_\parallel = \frac{2\pi}{L} N v_\parallel = \frac{2\pi N}{T}$ and, therefore,

$$e^{i\frac{\Omega T}{2}} \sin\frac{\Omega T}{2} = e^{i\left(1-\frac{v_\parallel}{c}\right)\frac{\omega T}{2}} \sin\left(\left(1-\frac{v_\parallel}{c}\right)\frac{\omega T}{2}\right)$$

for both Ω values. We also note that

$$1-\frac{v_\parallel}{c} = \frac{1-(v_\parallel/c)^2}{1+v_\parallel/c} \cong \frac{1}{2\gamma^2}$$

for $v_\parallel \lesssim c$ and $v_\perp \ll v_\parallel$, as specified. Then

$$\frac{dE(\omega)}{d\Omega} = \frac{\omega^2 e^2}{4\pi^2 c^3}\left(\frac{1}{2}v_\perp \sin\frac{\omega T}{4\gamma^2}\right)^2 \left|\frac{\vec{e}_y + i\vec{e}_x}{\frac{\omega}{4\gamma^2}+\frac{\pi N}{T}} + \frac{\vec{e}_y - i\vec{e}_x}{\frac{\omega}{4\gamma^2}-\frac{\pi N}{T}}\right|^2$$

$$= \frac{\omega^2 e^2}{8\pi^2 c^3}(v_\perp T)^2 (\sin\phi)^2 \left(\frac{1}{(\phi+N\pi)^2} + \frac{1}{(\phi-N\pi)^2}\right)$$

with

$$\phi = \frac{\omega T}{4\gamma^2}.$$

(c) Only $\omega > 0$ counts, so that the maximum is reached at $\phi = N\pi$, where $\sin\phi = 0$ and $\frac{\sin\phi}{\phi - N\pi} = (-1)^N$.
This gives

$$\omega_{max} = \frac{4\gamma^2}{T} N\pi$$

and

$$\left.\frac{dE(\omega)}{d\Omega}\right|_{\omega=\omega_{max}} = \frac{2}{c^3}\left(N\gamma^2 e v_\perp\right)^2.$$

Since $T \propto L \propto N$, $\omega_{max} = 2\gamma^2 k_0 v_{\parallel}$ does not depend on the winding number N, whereas $\left.\frac{dE(\omega)}{d\Omega}\right|_{\omega=\omega_{max}}$ is proportional to N^2.

(d) We need $\sin\phi = 0$, or $\phi = N\pi \pm \pi$, which is a fractional difference of

$$\frac{\Delta\phi}{\phi_{max}} = \frac{\Delta\omega}{\omega_{max}} = \frac{1}{N}.$$