PC4242, AY 2009/2010, Sem 2, Test 1

Question ....1/5....

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$\boxed{1}$ In the unprimed coordinate system of Ah Liam, the particle trajectory is described by the 4-vector (event)

$$\begin{pmatrix} ct \\ \vec{u}t \end{pmatrix}$$

and in Ah Beng's primed coordinate system we have

$$\begin{pmatrix} ct' \\ \vec{u}'t' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma \vec{v}^T/c \\ \gamma \vec{v}/c & \overleftrightarrow{1} + (\gamma - 1)\frac{\vec{v}\vec{v}^T}{v^2} \end{pmatrix} \begin{pmatrix} ct \\ \vec{u}t \end{pmatrix}$$

$$= \begin{pmatrix} \gamma ct \\ \gamma \vec{v}t + \vec{u}t \end{pmatrix} \qquad (\text{since } \vec{v}^T\vec{u} = 0)$$

which tells us that

$$\vec{u}' = \vec{v} + \frac{1}{\gamma}\vec{u} \qquad \text{with } \frac{1}{\gamma} = \sqrt{1 - (v/c)^2} .$$

We could get this result alternatively by Lorentz transforming the 4-velocity of the particle.

It follows that

$$u'^2 = v^2 + \frac{1}{\gamma^2}u^2 = v^2 + u^2 - (uv/c)^2$$

$$= c^2 - \frac{1}{c^2}\underbrace{(c^2 - v^2)}_{<0}\underbrace{(c^2 - u^2)}_{\leq 0} < c^2 ,$$

as it should be.

Question 2/5

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[2] We know from lecture (Section 3.3 of the notes) that

$$\delta \vec{E} = -\delta_{coor}\vec{E} - \frac{\delta\vec{v}}{c} \times \vec{B},$$

$$\delta \vec{B} = -\delta_{coor}\vec{B} + \frac{\delta\vec{v}}{c} \times \vec{E},$$

$$\delta(\vec{\nabla}\times\vec{A}) = -\delta_{coor}(\vec{\nabla}\times\vec{A}) + \frac{\delta\vec{v}}{c}\times\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi\right),$$

$$\delta\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi\right) = -\delta_{coor}\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi\right) - \frac{\delta\vec{v}}{c}\times(\vec{\nabla}\times\vec{A}),$$

which imply that

$$\begin{aligned}\delta\mathcal{L} = -\delta_{coor}\mathcal{L} + \frac{1}{4\pi}\Big[& -\left(\frac{\delta\vec{v}}{c}\times\vec{B}\right)\cdot\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi\right) \\ & + \vec{E}\cdot\left(-\frac{\delta\vec{v}}{c}\times(\vec{\nabla}\times\vec{A})\right) \\ & - \left(\frac{\delta\vec{v}}{c}\times\vec{E}\right)\cdot(\vec{\nabla}\times\vec{A}) \\ & - \vec{B}\cdot\left(\frac{\delta\vec{v}}{c}\times\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi\right)\right) \\ & - \vec{E}\cdot\left(-\frac{\delta\vec{v}}{c}\times\vec{B}\right) \\ & + \vec{B}\cdot\left(\frac{\delta\vec{v}}{c}\times\vec{E}\right)\Big] \\ = -\delta_{coor}\mathcal{L}. &\end{aligned}$$

That is: $\mathcal{L}$ is a Lorentz scalar density.

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$\boxed{3}$ For $v \ll c$, we have $\sqrt{c^2-v^2} = c - \frac{1}{2}\frac{v^2}{c}$,

so that $mc\left(c-\sqrt{c^2-v^2}\right) = \frac{1}{2}mv^2$, indeed.
The response of L to an infinitesimal change of $\vec{v}$ is

$$\delta L = -\vec{p}\cdot\delta\vec{v} + mc\,\frac{\vec{v}\cdot\delta\vec{v}}{\sqrt{c^2-v^2}}$$

which tells us the constraint

$$\vec{p} = \frac{mc\,\vec{v}}{\sqrt{c^2-v^2}} = \gamma m\vec{v} \qquad (*)$$

with $\gamma = 1/\sqrt{1-(v/c)^2}$. We regard $(*)$ as defining $\vec{v}$ in terms of $\vec{p}$ and need

$$p^2 = (mc)^2\,\gamma^2\,(v/c)^2 = (mc)^2(\gamma^2-1)$$

or $\gamma = \frac{1}{mc}\sqrt{(mc)^2+\vec{p}^{\,2}}$

in the reverse relation

$$\vec{v} = \frac{\vec{p}}{\gamma m} = \frac{c\vec{p}}{\sqrt{(mc)^2+\vec{p}^{\,2}}}.$$

Then

$$H(\vec{r},\vec{p}) = \left[+\vec{p}\cdot\vec{v} - mc^2\left(1-\frac{1}{\gamma}\right)\right]\Big|_{\vec{v}=\vec{p}/(\gamma m)}$$

$$= \frac{cp^2}{\sqrt{(mc)^2+\vec{p}^{\,2}}} - mc^2 + \frac{mc^3}{\sqrt{(mc)^2+\vec{p}^{\,2}}}$$

or

$$H(\vec{r},\vec{p}) = c\sqrt{(mc)^2+\vec{p}^{\,2}} - mc^2 ,$$

the familiar expression for the relativistic kinetic energy.

For an actual trajectory we have $\vec{v} = \frac{d\vec{r}}{dt}$, and then

$$W_{12} = \int_{t_2}^{t_1} dt \; mc^2 \left(1-\sqrt{1-(v/c)^2}\right)$$

$$= mc^2 (\Delta t - \Delta s)$$

is (rest energy) × (difference between the observer's lapse of time Δt and the lapse of proper time Δs).

$\boxed{4}$ We verify:

$$\int (d\vec{r}) \left[-\vec{d}\cdot\vec{\nabla}\,\delta(\vec{r})\right] = \int (d\vec{r})\, \vec{\nabla}\cdot\left(-\vec{d}\,\delta(\vec{r})\right) = 0,$$

$$\int (d\vec{r})\, \vec{r} \left[-\vec{d}\cdot\vec{\nabla}\,\delta(\vec{r})\right] = \int (d\vec{r})\, \vec{\nabla}\cdot\left[-\vec{d}\,\underbrace{\vec{r}\,\delta(\vec{r})}_{=0}\right] + \int (d\vec{r})\, \vec{d}\cdot\overleftrightarrow{1}\,\delta(\vec{r}) = \vec{d}.$$

The electrostatic potential is

$$\Phi(\vec{r}) = \int (d\vec{r}\,')\, \frac{1}{|\vec{r}-\vec{r}\,'|} \left[-\vec{d}\cdot\vec{\nabla}'\,\delta(\vec{r}\,')\right]$$

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$$= \int (d\vec{r}\,') \ \delta(\vec{r}\,') \ \vec{d}\cdot\vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= -\vec{d}\cdot\vec{\nabla} \int (d\vec{r}\,') \ \delta(\vec{r}\,') \frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= -\vec{d}\cdot\vec{\nabla}\frac{1}{r} = \frac{\vec{d}\cdot\vec{r}}{r^3} .$$

We use this version for the calculation of the electric field,

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) = \vec{d}\cdot\vec{\nabla}\vec{\nabla}\frac{1}{r}$$

$$= \frac{3\vec{r}\,\vec{r}\cdot\vec{d} - r^2\vec{d}}{r^5} - \frac{4\pi}{3}\delta(\vec{r})\,\vec{d} .$$

The contact term is needed to ensure that $\vec{\nabla}\vec{\nabla}\frac{1}{r}$ has the correct dyadic trace:

$$\mathrm{tr}\left\{\vec{\nabla}\vec{\nabla}\frac{1}{r}\right\} = \nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$$

$$\overset{!}{=} \mathrm{tr}\left\{\frac{3\vec{r}\,\vec{r} - r^2\overleftrightarrow{1}}{r^5} - \frac{4\pi}{3}\overleftrightarrow{1}\,\delta(\vec{r})\right\}$$

$$= \frac{3r^2 - 3r^2}{r^5} - \frac{4\pi}{3}\times 3\ \delta(\vec{r})$$

$$= -4\pi\,\delta(\vec{r}) .$$