

Write answers on this side of the paper only.

- In the unprimed coordinate system of Ah Liam, the particle trajectory is described by the 4-vector (event)

$$\begin{pmatrix} ct \\ \vec{u}t \end{pmatrix}$$

and in Ah Beng's primed coordinate system we have

$$\begin{aligned} \begin{pmatrix} ct' \\ \vec{u}'t' \end{pmatrix} &= \begin{pmatrix} \gamma & \gamma \vec{v}^T/c \\ \gamma \vec{v}/c & \mathbb{1} + (\gamma - 1) \frac{\vec{v}\vec{v}^T}{v^2} \end{pmatrix} \begin{pmatrix} ct \\ \vec{u}t \end{pmatrix} \\ &= \begin{pmatrix} \gamma ct \\ \gamma \vec{v}t + \vec{u}t \end{pmatrix} \quad (\text{since } \vec{v}^T \vec{u} = 0) \end{aligned}$$

which tells us that

$$\vec{u}' = \vec{v} + \frac{1}{\gamma} \vec{u} \quad \text{with } \frac{1}{\gamma} = \sqrt{1 - (v/c)^2}.$$

We could get this result alternatively by Lorentz transforming the 4-velocity of the particle.

It follows that

$$\begin{aligned} u'^2 &= v^2 + \frac{1}{\gamma^2} u^2 = v^2 + u^2 - (uv/c)^2 \\ &= c^2 - \underbrace{\frac{1}{c^2} (c^2 - v^2)}_{< 0} \underbrace{(c^2 - u^2)}_{\leq 0} < c^2, \end{aligned}$$

as it should be.

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2] We know from lecture (Section 3.3 of the notes) that

$$\delta \vec{E} = -\delta_{\text{coord}} \vec{E} - \frac{\delta \vec{v}}{c} \times \vec{B},$$

$$\delta \vec{B} = -\delta_{\text{coord}} \vec{B} + \frac{\delta \vec{v}}{c} \times \vec{E},$$

$$\delta(\vec{\nabla} \times \vec{A}) = -\delta_{\text{coord}}(\vec{\nabla} \times \vec{A}) + \frac{\delta \vec{v}}{c} \times \left(-\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi\right),$$

$$\delta\left(-\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi\right) = -\delta_{\text{coord}}\left(-\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi\right) - \frac{\delta \vec{v}}{c} \times (\vec{\nabla} \times \vec{A}),$$

which imply that

$$\delta \mathcal{L} = -\delta_{\text{coord}} \mathcal{L} + \frac{1}{4\pi} \left[ -\left(\frac{\delta \vec{v}}{c} \times \vec{B}\right) \cdot \left(-\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi\right) \right.$$

$$+ \vec{E} \cdot \left(-\frac{\delta \vec{v}}{c} \times (\vec{\nabla} \times \vec{A})\right)$$

$$- \left(\frac{\delta \vec{v}}{c} \times \vec{E}\right) \cdot (\vec{\nabla} \times \vec{A})$$

$$- \vec{B} \cdot \left(\frac{\delta \vec{v}}{c} \times \left(-\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi\right)\right)$$

$$- \vec{E} \cdot \left(-\frac{\delta \vec{v}}{c} \times \vec{B}\right)$$

$$+ \vec{B} \cdot \left(\frac{\delta \vec{v}}{c} \times \vec{E}\right) \left. \right]$$

$$= -\delta_{\text{coord}} \mathcal{L}.$$

That is:  $\mathcal{L}$  is a Lorentz scalar density.

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$$\boxed{3} \text{ For } v \ll c, \text{ we have } \sqrt{c^2 - v^2} = c - \frac{1}{2} \frac{v^2}{c},$$

so that  $mc(c - \sqrt{c^2 - v^2}) = \frac{1}{2} m v^2$ , indeed.  
The response of  $L$  to an infinitesimal change of  $\vec{v}$  is

$$\delta L = -\vec{p} \cdot \delta \vec{v} + mc \frac{\vec{v} \cdot \delta \vec{v}}{\sqrt{c^2 - v^2}}$$

which tells us the constraint

$$\vec{p} = \frac{mc \vec{v}}{\sqrt{c^2 - v^2}} = \gamma m \vec{v} \quad (*)$$

with  $\gamma = 1/\sqrt{1 - (v/c)^2}$ . We regard (\*) as defining  $\vec{v}$  in terms of  $\vec{p}$  and need

$$p^2 = (mc)^2 \gamma^2 (v/c)^2 = (mc)^2 (\gamma^2 - 1)$$

$$\text{or } \gamma = \frac{1}{mc} \sqrt{(mc)^2 + \vec{p}^2}$$

in the reverse relation

$$\vec{v} = \frac{\vec{p}}{\gamma m} = \frac{c \vec{p}}{\sqrt{(mc)^2 + \vec{p}^2}}.$$

Then

$$\begin{aligned} H(\vec{r}, \vec{p}) &= \left[ + \vec{p} \cdot \vec{v} - mc^2 \left(1 - \frac{1}{\gamma}\right) \right]_{\vec{v} = \vec{p}/(\gamma m)} \\ &= \frac{c p^2}{\sqrt{(mc)^2 + \vec{p}^2}} - mc^2 + \frac{mc^3}{\sqrt{(mc)^2 + \vec{p}^2}} \end{aligned}$$

