We use cylindrical coordinates (polar coordinates in the xy plane), that is

$$\left.\begin{aligned} x &= s\cos\varphi \\ y &= s\sin\varphi \\ z &= z \end{aligned}\right\} \text{ with } s>0,\ \varphi \text{ any } 2\pi \text{ interval}$$

for which $(d\vec{r}) = s\,ds\,d\varphi\,dz$ and

$$\frac{\partial}{\partial x} = \cos\varphi\frac{\partial}{\partial s} - \frac{1}{s}\sin\varphi\frac{\partial}{\partial\varphi}\,, \quad \frac{\partial}{\partial y} = \sin\varphi\frac{\partial}{\partial s} + \frac{1}{s}\cos\varphi\frac{\partial}{\partial\varphi}.$$

Then

$$\vec{j}(\vec{r},t) = I\cos(\omega t)\,\delta(s-a)\,\delta(z)\,(\vec{e}_y\cos\varphi - \vec{e}_x\sin\varphi).$$

(a) Since $\vec{\nabla}\cdot\vec{j} = 0$, as we see from

$$\vec{\nabla}\cdot\left[\delta(s-a)\,\delta(z)\,(\vec{e}_y\cos\varphi - \vec{e}_x\sin\varphi)\right]$$

$$= \delta(z)\Big[\Big(\cos\frac{\partial}{\partial s} - \frac{1}{s}\sin\varphi\frac{\partial}{\partial\varphi}\Big)(-\sin\varphi)$$

$$+\Big(\sin\varphi\frac{\partial}{\partial s} + \frac{1}{s}\cos\varphi\frac{\partial}{\partial\varphi}\Big)\cos\varphi\Big]\,\delta(s-a)$$

$$= \delta(z)\,\delta(s-a)\,\frac{1}{s}\Big(\sin\varphi\frac{\partial}{\partial\varphi}\sin\varphi + \cos\varphi\frac{\partial}{\partial\varphi}\cos\varphi\Big) = 0,$$

we have $\frac{\partial}{\partial t}\rho(\vec{r},t) = 0$, so that there is no charge density that accompanies the current density,

$$\rho(\vec{r},t) = 0.$$

It follows that the electric dipole moment and the electric quadrupole moment vanish,

$$\vec{d}(t) = 0 \,, \quad \overleftrightarrow{Q}(t) = 0 \,.$$

In (7.3.12), we thus need only the magnetic dipole moment,

$$\vec{\mu}(t) = \frac{1}{2c} \int (d\vec{r})\, \vec{r} \times \vec{j}(\vec{r}, t)$$

$$= \frac{1}{2c} I \cos(\omega t) \int (d\vec{r})\, \delta(s-a)\, \delta(z)\, s\, \vec{e}_z \,,$$

where $\vec{r} \times (\vec{e}_z \times \vec{r}) \Big|_{z=0} = r^2 \vec{e}_z \Big|_{z=0} = s^2 \vec{e}_z$ is used.

This gives

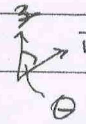
$$\vec{\mu}(t) = \frac{1}{2c} I \cos(\omega t)\, 2\pi a^2\, \vec{e}_z = \pi I \cos(\omega t) \frac{a^2}{c} \vec{e}_z$$

and then

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left| \vec{n} \times \ddot{\vec{\mu}}(t) \right|^2 = \frac{\pi}{4} \frac{I^2}{c} \left(\frac{a\omega}{c}\right)^4 \cos(\omega t)^2 \left| \vec{n} \times \vec{e}_z \right|^2$$

or, after averaging over one period, $\cos(\omega t)^2 \to \frac{1}{2}$

$$\frac{dP}{d\Omega} = \frac{\pi}{8} \frac{I^2}{c} \left(\frac{a\omega}{c}\right)^4 (\sin\theta)^2$$



and $P = \int d\Omega \frac{dP}{d\Omega} = \frac{\pi^2}{3} \frac{I^2}{c} \left(\frac{a\omega}{c}\right)^4$.

This applies when the ring antenna is small, that is $a \ll \lambda = c/\omega$.

Question. 3/5

Write answers on this side of the paper only.

(B) We use (7.2.4) where we need

$$\int (d\vec{r}\,') \frac{\partial}{\partial t} \vec{j}(\vec{r}\,', t_r)$$

$$= \frac{\partial}{\partial t_e} \int (d\vec{r}\,')\, I \cos(\omega t_e + \omega \vec{n}\cdot\vec{r}\,'/c)\, \delta(s'-a)\, \delta(z') \times (\vec{e}_y \cos\varphi' - \vec{e}_x \sin\varphi')$$

with $\vec{n} = \vec{e}_x \sin\Theta \cos\phi + \vec{e}_y \sin\Theta \sin\phi + \vec{e}_z \cos\Theta$, so that

$$\vec{n}\cdot\vec{r}\,' = s' \sin\Theta \cos(\varphi' - \phi) .$$

After shifting the azimuth φ' by ϕ: $\varphi' \to \varphi' + \phi$ in the integrand, this gives

$$\int (d\vec{r}\,') \frac{\partial}{\partial t} \vec{j}(\vec{r}\,'; t_r)$$

$$= \frac{\partial}{\partial t_e} Ia \int_{(2\pi)} d\varphi' \cos(\omega t_e + \frac{a\omega}{c} \sin\Theta \cos\varphi') \left(\vec{e}_y \cos(\varphi'+\phi) - \vec{e}_x \sin(\varphi'+\phi)\right)$$

where we can replace $\cos(\varphi'+\phi) \to \cos\varphi' \cos\phi$

$\sin(\varphi'+\phi) \to \cos\varphi' \sin\phi$

because the dropped terms are proportional to $\sin\varphi'$ and do not contribute to the periodic φ' integral. We so arrive at

$$\int (d\vec{r}\,') \frac{\partial}{\partial t} \vec{j}(\vec{r}\,'; t_r) = Ia (\vec{e}_y \cos\phi - \vec{e}_y \sin\phi)$$

$$\times \frac{\partial}{\partial t_e} \int_{(2\pi)} d\varphi \cos\varphi \cos(\omega t_e + \frac{\omega a}{c} \sin\Theta \cos\varphi) .$$

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The integral over φ is equal to

$$\mathrm{Re} \int_{(2\pi)} d\varphi \cos\varphi \; e^{i\omega t_e} \, e^{i \frac{a\omega}{c} \sin\theta \cos\varphi}$$

$$= \mathrm{Re}\, e^{i\omega t_e} \int_{(2\pi)} d\varphi \, \tfrac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) e^{i \frac{a\omega}{c} \sin\theta \cos\varphi}$$

$$= \mathrm{Re}\, e^{i\omega t_e} \, i\pi \left[J_1\left(\tfrac{a\omega}{c} \sin\theta\right) - J_{-1}\left(\tfrac{a\omega}{c}\sin\theta\right)\right]$$

$$= -2\pi \sin(\omega t_e) \, J_1\left(\tfrac{a\omega}{c} \sin\theta\right).$$

Then, with $\left|\vec{n} \times (\vec{e}_y \cos\phi - \vec{e}_x \sin\phi)\right|^2 = 1$ because the two unit vectors are orthogonal, we get

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left[2\pi I a\omega \cos(\omega t_e) \, J_1\left(\tfrac{a\omega}{c} \sin\theta\right)\right]^2$$

or

$$\frac{dP}{d\Omega} = \frac{\pi}{2} \frac{I^2}{c} \left(\frac{a\omega}{c}\right)^2 J_1\left(\frac{a\omega}{c}\sin\theta\right)^2$$

after averaging over one period.

For $\frac{a\omega}{c} \ll 1$, we have $J_1\left(\frac{a\omega}{c}\sin\theta\right) = \frac{1}{2}\frac{a\omega}{c}\sin\theta$ and recover the result of (a),

$$\frac{dP}{d\Omega} = \frac{\pi}{8} \frac{I^2}{c} \left(\frac{a\omega}{c}\right)^4 (\sin\theta)^2,$$

as we should.

Question....5/5.......

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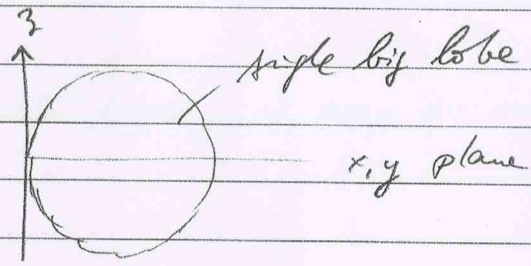
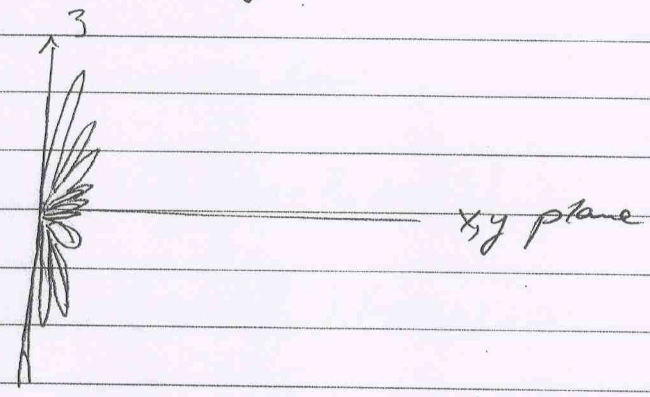
(c) For $\frac{a\omega}{c} \ll 1$, we have the familiar pattern of dipole radiation:



For $\frac{a\omega}{c} \gg 1$, the argument of the Bessel function covers the range $0 \ldots a\omega/c$ and there are many zeros and extrema of the Bessel function in this range. Accordingly, we now get many lobes with less intensity where $\frac{a\omega}{c}\sin\theta$ is large,



Picture should be symmetric to the xy plane because the sign of $\sin\theta$ does not matter.