

Question.. 1/5.....

Do not write on either margin

Write answers on this side of the paper only.

We use cylindrical coordinates (polar coordinates in the xy plane), that is

$$\left. \begin{aligned} x &= s \cos \varphi \\ y &= s \sin \varphi \\ z &= z \end{aligned} \right\} \text{with } s > 0, \varphi \text{ any } 2\pi \text{ interval}$$

for which $(d\vec{r}) = s ds d\varphi dz$ and

$$\frac{\partial}{\partial x} = \cos \varphi \frac{\partial}{\partial s} - \frac{1}{s} \sin \varphi \frac{\partial}{\partial \varphi}, \quad \frac{\partial}{\partial y} = \sin \varphi \frac{\partial}{\partial s} + \frac{1}{s} \cos \varphi \frac{\partial}{\partial \varphi}.$$

Then

$$\vec{j}(\vec{r}, t) = I \cos(\omega t) \delta(s-a) \delta(z) (\vec{e}_y \cos \varphi - \vec{e}_x \sin \varphi).$$

(a) Since $\vec{\nabla} \cdot \vec{j} = 0$, as we see from

$$\begin{aligned} & \vec{\nabla} \cdot [\delta(s-a) \delta(z) (\vec{e}_y \cos \varphi - \vec{e}_x \sin \varphi)] \\ &= \delta(z) \left[\left(\cos \frac{\partial}{\partial s} - \frac{1}{s} \sin \varphi \frac{\partial}{\partial \varphi} \right) (-\sin \varphi) \right. \\ & \quad \left. + \left(\sin \varphi \frac{\partial}{\partial s} + \frac{1}{s} \cos \varphi \frac{\partial}{\partial \varphi} \right) \cos \varphi \right] \delta(s-a) \\ &= \delta(z) \delta(s-a) \frac{1}{s} (\sin \varphi \frac{\partial}{\partial \varphi} \sin \varphi + \cos \varphi \frac{\partial}{\partial \varphi} \cos \varphi) = 0, \end{aligned}$$

We have $\frac{\partial}{\partial t} \rho(\vec{r}, t) = 0$, so that there is no charge density that accompanies the current density,

$$\rho(\vec{r}, t) = 0.$$

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It follows that the electric dipole moment and the electric quadrupole moment vanish,

$$\vec{d}(t) = 0, \quad \vec{Q}(t) = 0.$$

In (7.3.12), we thus need only the magnetic dipole moment,

$$\begin{aligned} \vec{\mu}(t) &= \frac{1}{2c} \int (d\vec{r}) \vec{r} \times \vec{j}(\vec{r}, t) \\ &= \frac{1}{2c} I \cos(\omega t) \int (d\vec{r}) \delta(r-a) \delta(z) s \vec{e}_z, \end{aligned}$$

where $\vec{r} \times (\vec{e}_z \times \vec{r}) \Big|_{z=0} = r^2 \vec{e}_z \Big|_{z=0} = s^2 \vec{e}_z$ is used.

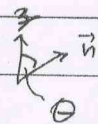
This gives

$$\vec{\mu}(t) = \frac{1}{2c} I \cos(\omega t) 2\pi a^2 \vec{e}_z = \pi I \cos(\omega t) \frac{a^2}{c} \vec{e}_z$$

and then

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left| \vec{n} \times \ddot{\vec{\mu}}(t) \right|^2 = \frac{\pi}{4} \frac{I^2}{c} \left(\frac{a\omega}{c} \right)^4 \cos^2(\omega t) \left| \vec{n} \times \vec{e}_z \right|^2$$

or, after averaging over one period, $\cos^2(\omega t) \rightarrow \frac{1}{2}$

$$\frac{dP}{d\Omega} = \frac{\pi}{8} \frac{I^2}{c} \left(\frac{a\omega}{c} \right)^4 (\sin\theta)^2$$


$$\text{and } P = \int d\Omega \frac{dP}{d\Omega} = \frac{\pi^2}{3} \frac{I^2}{c} \left(\frac{a\omega}{c} \right)^4.$$

This applies when the ring antenna is small, that is $a \ll \lambda = c/\omega$.

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(B) We use (7.2.4) where we need

$$\int (d\vec{r}') \frac{\partial}{\partial t} \vec{j}(\vec{r}', t_r)$$

$$= \frac{\partial}{\partial t_e} \int (d\vec{r}') I \cos(\omega t_e + \omega \vec{n} \cdot \vec{r}' / c) \delta(r' - a) \delta(z')$$

$$\times (\vec{e}_y \cos \varphi' - \vec{e}_x \sin \varphi')$$

with $\vec{n} = \vec{e}_x \sin \theta \cos \phi + \vec{e}_y \sin \theta \sin \phi + \vec{e}_z \cos \theta$, so that

$$\vec{n} \cdot \vec{r}' = r' \sin \theta \cos(\varphi' - \phi).$$

After shifting the azimuth φ' by ϕ : $\varphi' \rightarrow \varphi' + \phi$ in the integrand, this gives

$$\int (d\vec{r}') \frac{\partial}{\partial t} \vec{j}(\vec{r}', t_r)$$

$$= \frac{\partial}{\partial t_e} I a \int_{(2\pi)} d\varphi' \cos(\omega t_e + \frac{\omega a}{c} \sin \theta \cos \varphi') (\vec{e}_y \cos(\varphi' + \phi)$$

$$- \vec{e}_x \sin(\varphi' + \phi))$$

where we can replace $\cos(\varphi' + \phi) \rightarrow \cos \varphi' \cos \phi$
 $\sin(\varphi' + \phi) \rightarrow \cos \varphi' \sin \phi$ because the dropped terms are proportional to $\sin \varphi'$ and do not contribute to the periodic φ' integral.

We so arrive at

$$\int (d\vec{r}') \frac{\partial}{\partial t} \vec{j}(\vec{r}', t_r) = I a (\vec{e}_y \cos \phi - \vec{e}_x \sin \phi)$$

$$\times \frac{\partial}{\partial t_e} \int_{(2\pi)} d\varphi \cos \varphi \cos(\omega t_e + \frac{\omega a}{c} \sin \theta \cos \varphi).$$

