PC4242, AY 2009/10, Sem 2, Exam

Question 1/5

Write answers on this side of the paper only.

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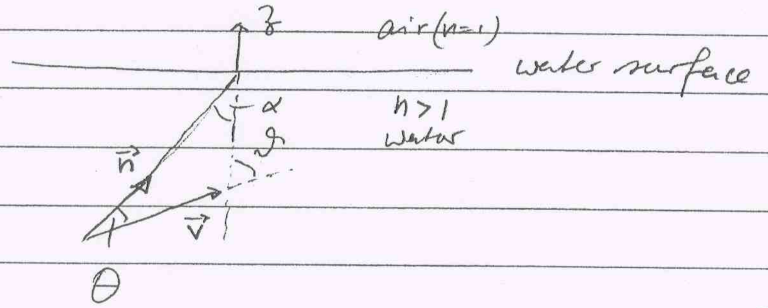
1 Rayleigh scattering differs from Thomson scattering only by the frequency dependent factor, so that

$$\left(\frac{d\sigma}{d\Omega}\right)_{Ray} = \left(\frac{d\sigma}{d\Omega}\right)_{Th} \left|\frac{\omega^2}{\omega_0^2-\omega^2-i\gamma\omega}\right|^2$$

$$= r_0^2 \frac{1}{2}\left(1+(\cos\theta)^2\right) \frac{\omega^4}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2} .$$

2
(a)




The angle of incident α must be less than the critical angle for total internal reflection,

$$\sin\alpha < \frac{1}{n} = \sin\alpha_0 .$$

The angle ϑ between $\vec{v}$ of the electron and the normal surface vector $\vec{e}_z$ is denoted by ϑ, so that Cherenkov radiation can get across the surface if

$$\vartheta = \theta + \alpha < \theta + \alpha_0$$

where $\cos\theta = \frac{c}{nv}$ (Cherenkov) and $\sin\alpha_0 = \frac{1}{n}$ (Snell).

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(b) We would need $\theta + \alpha_0 \geq \frac{\pi}{2}$, but in fact we have

$$\cos(\theta + \alpha_0) = \cos\theta \cos\alpha_0 - \sin\theta \sin\alpha_0$$

$$= \frac{c}{nv}\sqrt{1 - \frac{1}{n^2}} - \frac{1}{n}\sqrt{1 - \left(\frac{c}{nv}\right)^2}$$

$$= \frac{1}{n^2 v}\left(\sqrt{(nc)^2 - c^2} - \sqrt{(nv)^2 - c^2}\right) > 0$$

which tells us that $\theta + \alpha_0 < \frac{\pi}{2}$. Answer: <u>No.</u>

$\boxed{3}$ (a) The retarded time (= emission time corrected for internal retardation) for the $l$th ring antenna ($l = 0, \pm 1, \pm 2, \dots, \pm M$) is

$$t_r = \underbrace{t - \frac{r}{c} + \frac{1}{c}\vec{n}\cdot\vec{r}\,'}_{\text{central antenna with } l=0} - \frac{1}{c}\,\vec{n}\cdot \underbrace{lD\vec{e}_z}_{\substack{\text{displacement}\\ \text{of the } l\text{th}\\ \text{antenna}}}$$

$$= \left(t - \frac{r}{c} - l\frac{D}{c}\cos\theta\right) + \frac{1}{c}\vec{n}\cdot\vec{r}\,'.$$

Effectively we have the replacement

$$t_e = t - \frac{r}{c} \longrightarrow t_e - l\frac{D}{c}\cos\theta.$$

The integrated time derivation of the current has a time-dependence given by

$\cos(\omega t_e + \alpha)$

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for the central antenna ($l=0$) whereby $\alpha$ is some irrelevant phase that will not survive the averaging over one period. Accordingly,

$$\left(\frac{dP}{d\Omega}\right)_N = \left(\frac{dP}{d\Omega}\right)_1 \quad 2\left(\sum_{l=-M}^{M} \cos\left(\omega t_e + \alpha - l\,\frac{\omega D}{c}\cos\theta\right)\right)^2$$

$\uparrow$ all N rings $\qquad$ $\uparrow$ central ring only

where the sum is a geometric summation,

$$\sum_{l=-M}^{M} \cos\left(\omega t_e + \alpha - l\,\frac{\omega D}{c}\cos\theta\right)$$

$$= \mathrm{Re}\; e^{i(\omega t_e + \alpha)} \sum_{l=-M}^{M} e^{-i l \frac{\omega D}{c}\cos\theta}$$

$$= \cos(\omega t_e + \alpha)\; \frac{\sin\left((2M+1)\frac{\omega D}{2c}\cos\theta\right)}{\sin\left(\frac{\omega D}{2c}\cos\theta\right)}.$$

With $N = 2M+1$ and $\frac{\omega D}{2c} = \pi\frac{D}{\lambda}$, then,

$$\left(\frac{dP}{d\Omega}\right)_N = \left(\frac{dP}{d\Omega}\right)_1 \left(\frac{\sin\left(\pi N \frac{D}{\lambda}\cos\theta\right)}{\sin\left(\pi\frac{D}{\lambda}\cos\theta\right)}\right)^2$$

$$= \underbrace{\frac{\pi}{2}\frac{I^2}{c}\left(\frac{a\omega}{c}\right)^2 J_1\left(\frac{a\omega}{c}\sin\theta\right)^2}_{\text{one ring antenna}} \underbrace{\left(\frac{\sin\left(\pi N\frac{D}{\lambda}\cos\theta\right)}{\sin\left(\pi\frac{D}{\lambda}\cos\theta\right)}\right)^2}_{\text{interference of all N antennas}}.$$

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(b) The interference enhances the radiation emitted into the xy plane where $\cos\theta = 0$, and the interference term equals $N^2$.
There is destructive interference for those directions with

$$N \frac{D}{\lambda} \cos\theta = \pm 1, \pm 2, \dots$$

We get additional suppression of radiation in the z direction (on top of the vanishing of $J_1(\dots)$ for $\sin\theta = 0$) when $N\frac{D}{\lambda}$ is integer.

$\boxed{4}$ In the relativistic energy-loss formula

$$-\frac{dE}{dt} = \frac{2e^2}{3c^3}\left[1-\left(\frac{v(t)}{c}\right)^2\right]^{-3}\left[\left(\frac{d\vec{v}}{dt}\right)^2 - \left(\frac{\vec{v}}{c}\times\frac{d\vec{v}}{dt}\right)^2\right]$$

we insert

$$\frac{d}{dt}\vec{v} = -\frac{\vec{v}_0}{T}, \quad \vec{v}(t) = \left(1-\frac{t}{T}\right)\vec{v}_0,$$

valid while the stopping process lasts.
This give the total radiated energy

$$E_{rad} = \frac{2e^2}{3c^3}\int_0^T dt \left[1-\left(\frac{v_0}{c}\right)^2\left(1-\frac{t}{T}\right)^2\right]^{-3}\left(\frac{v_0}{T}\right)^2$$

or, after the substitution

$$\frac{v_0}{c}\left(1-\frac{t}{T}\right) = \tanh\vartheta,$$

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$$E_{rad} = \frac{2}{3}\frac{e^2}{cT}\left(\frac{v_0}{c}\right)^2 \int_0^{\Theta} d\vartheta\, (\cosh\vartheta)^4$$

where $\Theta$ is the rapidity associated with $v_0$, $\frac{v_0}{c} = \tanh\Theta$. Using $(\cosh\vartheta)^2 = \frac{1}{2} + \frac{1}{2}\cosh(2\vartheta)$ twice, we get

$$E_{rad} = \frac{2}{3}\frac{e^2}{cT}\left(\frac{v_0}{c}\right)^2 \left[\frac{3}{8}\Theta + \frac{1}{4}\sinh(2\Theta) + \frac{1}{32}\sinh(4\Theta)\right]$$

$$= \frac{2}{3}\frac{e^2}{cT}\left(\frac{v_0}{c}\right)^2 \left[\frac{3}{8}\Theta + \frac{3}{8}\sinh\Theta\cosh\Theta + \frac{1}{4}\sinh\Theta(\cosh\Theta)^3\right]$$

$$= \frac{2}{3}\frac{e^2}{cT}\beta_0^2\left(\frac{3}{8}\Theta + \frac{3}{8}\beta_0\gamma_0^2 + \frac{1}{4}\beta_0\gamma_0^4\right)$$

with $\beta_0 = \frac{v_0}{c}$, $\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}} = \cosh\Theta$.

(b) When $\beta_0 \lesssim 1$, $\gamma_0 \gg 1$, the last term in the parentheses dominates the others, and we have

$$E_{rad} \simeq \frac{2}{3}\frac{e^2}{cT}\frac{1}{4}\gamma_0^4 = \frac{1}{6}\frac{e^2}{cT}\gamma_0^4.$$

This can also be derived by realizing that $(\cosh\vartheta)^4 \simeq \left(\frac{1}{2}e^{\vartheta}\right)^4 = \frac{1}{4}\frac{\partial}{\partial\vartheta}\left(\frac{1}{2}e^{\vartheta}\right)^4$ in the integral at the top of the page, under these circumstances.