Do not write on either margin

1 Rotate by $90°$ about x-axis: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ -z \\ y \end{pmatrix}$;

rotate by $180°$ about y-axis: $\begin{pmatrix} x \\ -z \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ -z \\ -y \end{pmatrix}$;

rotate by $90°$ about z-axis: $\begin{pmatrix} -x \\ -z \\ -y \end{pmatrix} \to \begin{pmatrix} z \\ -x \\ -y \end{pmatrix}$;

overall rotation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} z \\ -x \\ -y \end{pmatrix},$$

so that all vectors with components $\begin{pmatrix} a \\ -a \\ a \end{pmatrix}$ are unchanged by the rotation. It follows that we can regard the unit vector

$$\vec{e} \mathrel{\hat{=}} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

as specifying the axis of rotation; equivalently, this role can be played by $-\vec{e}$.

We take a vector perpendicular to $\vec{e}$, such as the one with components $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ to find the rotation angle. First, the net effect is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix},$$

then the cosine of the rotation angle is found from $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = 2\cos\phi$,

which gives $\cos\phi = -\frac{1}{2}$, so that $\phi = 120°$ or $-120°$. Finally, since

Question 2

Write answers on this side of the paper only.

Do not write on either margin

$$\left[\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\-1\\-1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix} = -3<0\,,$$

the rotation around axis $\vec{e}$ is by $-120°$; or, equivalently, the rotation around axis $-\vec{e}$ is by $120°$.

$\boxed{2}$ We use the retarded Green's function for the harmonic oscillator without damping ($\gamma=0$, $\omega=\omega_0$), and so have

$$x(t) = \int_{-\infty}^{t} dt' \, \frac{1}{\omega_0}\sin(\omega_0(t-t'))\,\frac{F(t')}{m}\,,$$

which clearly has $x(t)=0$ for all times before the period during which $F(t)\neq 0$. The velocity is

$$\dot{x}(t) = \int_{-\infty}^{t} dt' \cos(\omega_0(t-t'))\,\frac{F(t')}{m}\,,$$

and the time-dependence of the force must then be such that

$$\int_{-\infty}^{t} dt' \sin(\omega_0(t-t'))F(t') = 0$$

and

$$\int_{-\infty}^{t} dt' \cos(\omega_0(t-t'))\,F(t') = 0$$

for all times t after the period during which $F(t)\neq 0$. Taken together these equations tell us that

Do not write on either margin

$$\int_{-\infty}^{t} dt' \, e^{i\omega_0(t-t')} F(t') = 0$$

for all late times, but then $F=0$ for even late times, so that we arrive at the compact statement

$$\int_{-\infty}^{\infty} dt \, e^{-i\omega_0 t} F(t) = 0.$$

That is: The Fourier component of $F(t)$ at the circular frequency of the oscillator must vanish, but other than that there are no further restrictions on $F(t)$.

$\boxed{3}$ The components of the force are

$$F_x = -\frac{\partial}{\partial x} V(x,y) = k_1 V_0 \sin(k_1 x)\cos(k_2 y),$$

$$F_y = -\frac{\partial}{\partial y} V(x,y) = k_2 V_0 \cos(k_1 x)\sin(k_2 y).$$

Therefore we have two kinds of points where $F_x = 0$ and $F_y = 0$, namely

(i) $\sin(k_1 x) = 0$ and $\sin(k_2 y) = 0$,

(ii) $\cos(k_1 x) = 0$ and $\cos(k_2 y) = 0$.

Question 4

Write answers on this side of the paper only.

Do not write on either margin

The 2x2 matrix of second derivatives,

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) V(x,y) = V_0 \begin{pmatrix} -k_1^2 \cos(k_1 x)\cos(k_2 y) & k_1 k_2 \sin(k_1 x)\sin(k_2 y) \\ k_1 k_2 \sin(k_1 x)\sin(k_2 y) & -k_2^2 \cos(k_1 x)\cos(k_2 y) \end{pmatrix}$$

is

$$\pm k_1 k_2 V_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for extrema of kind (ii) with $\sin(k_1 x)\sin(k_2 y) = \pm 1$; for both sign the 2x2 matrix has eigenvalues $\pm k_1 k_2 V_0$, so that these are <u>saddle points</u>

For extrema of kind (i) with $\cos(k_1 x)\cos(k_2 y) = \pm 1$, we have the 2x2 matrix

$$\mp V_0 \begin{pmatrix} k_1^2 & 0 \\ 0 & k_2^2 \end{pmatrix} \text{ with eigenvalues } \mp V_0 k_1^2, \mp V_0 k_2^2,$$

so that we have local maxima for the upper sign and local minima for the lower sign.

At the local minima the periods of the small-amplitude oscillations are, therefore, given by

$$\frac{2\pi}{k_1}\sqrt{\frac{m}{V_0}} \quad \text{and} \quad \frac{2\pi}{k_2}\sqrt{\frac{m}{V_0}}.$$

Question ..... 5 .....

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{4}$ We recall that

$$\vec{\nabla} = \vec{e}_s \frac{\partial}{\partial s} + \vec{e}_\varphi \frac{1}{s}\frac{\partial}{\partial \varphi} + \vec{e}_z \frac{\partial}{\partial z}$$

where $\vec{e}_s$ and $\vec{e}_\varphi$ depend on φ:

$$\frac{\partial}{\partial \varphi} \vec{e}_s = \vec{e}_\varphi \, , \quad \frac{\partial}{\partial \varphi} \vec{e}_\varphi = -\vec{e}_s \, .$$

Therefore,

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial s} A_s + \frac{1}{s} A_s + \frac{1}{s}\frac{\partial}{\partial \varphi} A_\varphi + \frac{\partial}{\partial z} A_z$$

$$= \frac{1}{s}\frac{\partial}{\partial s}(s A_s) + \frac{1}{s}\frac{\partial}{\partial \varphi} A_\varphi + \frac{\partial}{\partial z} A_z \, .$$

For $\vec{A} = x\vec{e}_x = s\cos\varphi(\vec{e}_s \cos\varphi - \vec{e}_\varphi \sin\varphi)$, that is: $A_s = s(\cos\varphi)^2$, $A_\varphi = -s\sin\varphi\cos\varphi$, $A_z = 0$, this gives

$$\vec{\nabla} \cdot \vec{A} = 2(\cos\varphi)^2 - (\cos\varphi)^2 + (\sin\varphi)^2 = 1 \, ,$$

consistent with $\vec{\nabla} \cdot (x\vec{e}_x) = \frac{\partial}{\partial x} x = 1$.

For $\vec{A} = \vec{r} = s\vec{e}_s + z\vec{e}_z$, that is: $A_s = s$, $A_\varphi = 0$, $A_z = z$, we get

$$\vec{\nabla} \cdot \vec{r} = 2 + 0 + 1 = 3 \, ,$$

consistent with

$$\vec{\nabla} \cdot \vec{r} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3 .$$