AY 2011/12, Sem I, PC2132, Exam

Question. . . 1/6. . . . .

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[1] The curl of $\vec{F}$ is $\vec{\partial}\times\vec{F} = \vec{\nabla}\times(m\omega^2\vec{r} - m\vec{\omega}\,\vec{\omega}\cdot\vec{r}) = m\vec{\omega}\times\vec{\omega} = 0$, so, yes, $\vec{F}$ is conservative, and since $\vec{F}$ is linear in $\vec{r}$, the potential energy is $-\frac{1}{2}\vec{r}\cdot\vec{F} = \frac{m}{2}\vec{r}\cdot[\vec{\omega}\times(\vec{\omega}\times\vec{r})]$

$$= -\frac{m}{2}(\vec{\omega}\times\vec{r})^2.$$

[2] (a) The second derivative of the potential energy is

$$V''(x) = -\frac{d}{dx}F(x) = 2ax,$$

so that $V''(\pm x_0) = \pm 2ax_0$. It follows that there is a stable equilibrium at $x = x_0$ and an unstable equilibrium at $x = -x_0$.

(b) For $x \cong x_0$ we have $m(\frac{d}{dt})^2(x-x_0) = F(x_0 + (x-x_0))$

$$\cong F'(x_0)(x-x_0)$$

or

$$m(\frac{d}{dt})^2(x-x_0) \cong -2ax_0(x-x_0) = -m\omega^2(x-x_0)$$

with $\omega = \sqrt{2ax_0/m}$ so that the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2ax_0}}.$$

(c) The period will be longer because it is very long when the energy is close to that required to get to the unstable equilibrium point at $x = -x_0$.

[3] (a) From

$$L = \frac{m}{2}\vec{v}^2 - V(\vec{r}) + \vec{v}\cdot\vec{\nabla}u(\vec{r})$$

we have

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \vec{v}} - \frac{\partial L}{\partial \vec{r}} = \frac{d}{dt}[m\vec{v} + \vec{\nabla}u(\vec{r})] - [-\vec{\nabla}V(\vec{r}) + \vec{v}\cdot\vec{\nabla}\,\vec{\nabla}u(\vec{r})]$$

$$= m\frac{d}{dt}\vec{v} + \underline{\vec{v}\cdot\vec{\nabla}\,\vec{\nabla}u(\vec{r})} + \vec{\nabla}V(\vec{r}) - \underline{\vec{v}\cdot\vec{\nabla}\,\vec{\nabla}u(\vec{r})}$$

or

$$m\frac{d}{dt}\vec{v} = -\vec{\nabla}V(\vec{r}),$$

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which does not contain any trace of $u(\vec{r})$.

(b) The Hamilton function is, for $\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \vec{\nabla} u(\vec{r})$,

$$H = \left(\vec{v}\cdot\frac{\partial L}{\partial \vec{v}} - L\right)\Bigg|_{\vec{v} = [\vec{p} - \vec{\nabla} u(\vec{r})]/m}$$

$$= \frac{1}{2m}\left[\vec{p} - \vec{\nabla} u(\vec{r})\right]^2 + V(\vec{r}).$$

It depends on the choice of $u(\vec{r})$ because $\vec{\nabla} u$ enters the relation between $\vec{p}$ and $\vec{v}$.

$\boxed{4}$ The body is composed of a homogeneous ball of radius R and an ellipsoid with half-axes $2R, 2R, R$, both having mass density ρ_0. The total mass is, therefore,

$$M = \frac{4\pi}{3}\rho_0 R^3 + \frac{4\pi}{3}\rho_0\, 4R^3 = \frac{4\pi}{3}\rho_0\, 5R^3,$$

so that

$$\rho_0 = \frac{3}{20\pi}\frac{M}{R^3}.$$

(a) In $\overleftrightarrow{I} = \int (d\vec{r})\, \rho(\vec{r})\, (r^2 \overleftrightarrow{1} - \vec{r}\vec{r})$, we have

$$\int (d\vec{r})\, \rho(\vec{r})\, \vec{r}\vec{r} = \rho_0 \underset{(\text{ball})}{\int} (d\vec{r})\, \vec{r}\vec{r} + \rho_0 \underset{(\text{ellipsoid})}{\int} (d\vec{r})\, \vec{r}\vec{r} = \overleftrightarrow{B} + \overleftrightarrow{E}$$

with

$$\overleftrightarrow{B} = \rho_0 \int_{r<R} (d\vec{r})\, \vec{r}\vec{r} = \rho_0 \frac{1}{3}\overleftrightarrow{1} \int_{r<R} (d\vec{r})\, r^2 = \rho_0 \frac{4\pi}{3}\frac{R^5}{5}\overleftrightarrow{1}$$

and (recall Exercise 38)

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$$\overleftrightarrow{E} = \rho_0 \int (d\vec{r})\, \vec{r}\,\vec{r}\; \eta\left(1 - \frac{x^2+y^2}{4R^2} - \frac{z^2}{R^2}\right)$$

$x = 2x'$
$y = 2y'$
$z = z'$

$$\stackrel{\downarrow}{=} \rho_0 4 \int_{r'<R} (d\vec{r}')\, \left(4x'^2\, \vec{e}_x\vec{e}_x + 4y'^2\, \vec{e}_y\vec{e}_y + z'^2\, \vec{e}_z\vec{e}_z\right)$$

(terms like $xy\,\vec{e}_x\vec{e}_y$ are odd and do not contribute)

$$= \rho_0\, \frac{4}{3} \int_{r'<R} (d\vec{r}')\, r'^2 \left(4\vec{e}_x\vec{e}_x + 4\vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z\right)$$

$$= \rho_0\, \frac{4\pi}{3} \cdot \frac{4R^5}{5} \left(4\overleftrightarrow{1} - 3\vec{e}_z\vec{e}_z\right).$$

Together,

$$\overleftrightarrow{B} + \overleftrightarrow{E} = \frac{4\pi}{3}\rho_0 \frac{R^5}{5}\left(17\overleftrightarrow{1} - 12\,\vec{e}_z\vec{e}_z\right) = \frac{MR^2}{25}\left(17\overleftrightarrow{1} - 12\,\vec{e}_z\vec{e}_z\right)$$

so that

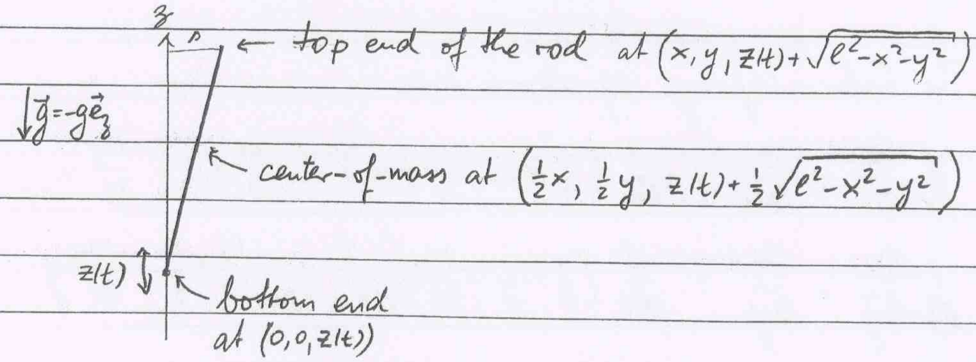
$$\overleftrightarrow{I} = \overleftrightarrow{1}\,\mathrm{tr}\{\overleftrightarrow{B} + \overleftrightarrow{E}\} - (\overleftrightarrow{B} + \overleftrightarrow{E})$$
$$= \frac{MR^2}{25}\left[29\overleftrightarrow{1} - (17\overleftrightarrow{1} - 12\vec{e}_z\vec{e}_z)\right] = \frac{MR^2}{25}\left(22\overleftrightarrow{1} + 12\vec{e}_z\vec{e}_z\right).$$

(b) The angular momentum is

$$\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega} = \frac{MR^2}{25}\left[22\,\vec{\omega} + 12\,\vec{e}_z \underbrace{\vec{e}_z\cdot\vec{\omega}}_{\omega\cos\theta}\right]$$

$$= \frac{MR^2}{25}\,\omega\left(22\,\vec{e}_x \sin\theta + 34\,\vec{e}_z \cos\theta\right).$$

$\boxed{5}$



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A mass element of the rod at $(ux, uy, z(t) + u\sqrt{l^2-x^2-y^2})$, $0 \le u \le 1$, has velocity $(u\dot{x}, u\dot{y}, \dot{z} - u(x\dot{x}+y\dot{y})/\sqrt{l^2-x^2-y^2})$, so that the kinetic energy is

$$\int_0^1 du \, \frac{1}{2} M \left(u^2\dot{x}^2 + u^2\dot{y}^2 + \dot{z}^2 - 2u\dot{z}\frac{x\dot{x}+y\dot{y}}{\sqrt{l^2-x^2-y^2}} + u^2 \frac{(x\dot{x}+y\dot{y})^2}{l^2-x^2-y^2} \right)$$

$$= \frac{M}{6}\left(\dot{x}^2 + \dot{y}^2 + 3\dot{z}^2 - 3\dot{z}\frac{x\dot{x}+y\dot{y}}{\sqrt{l^2-x^2-y^2}} + \frac{(x\dot{x}+y\dot{y})^2}{l^2-x^2-y^2} \right)$$

$$= \frac{M}{6}\left[\dot{x}^2 + \dot{y}^2 - \frac{3\dot{z}}{l}(x\dot{x}+y\dot{y}) \right] + \cdots ,$$

where the ellipsis stands for the terms of higher than quadratic order in $x, \dot{x}, y, \dot{y}$ and for the $\dot{z}^2$ term which is independent of these coordinates.

We combine this with the potential energy

$$Mg\left(z(t) + \frac{1}{2}\sqrt{l^2-x^2-y^2} \right) = -\frac{1}{4} Mg \frac{x^2+y^2}{l} + \cdots$$

to the Lagrange function for motion restricted to small values of $s = \sqrt{x^2+y^2}$, that is $s \ll l$,

$$L = \frac{M}{6}(\dot{x}^2+\dot{y}^2) - \frac{M}{2l}\dot{z}(x\dot{x}+y\dot{y}) + \frac{Mg}{4l}(x^2+y^2).$$

The Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} : \quad \frac{d}{dt}\left(\frac{1}{3}M\dot{x} - \frac{M}{2l}\dot{z}x \right) = -\frac{M}{2l}\dot{z}\dot{x} + \frac{Mg}{2l}x ,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y} : \quad \frac{d}{dt}\left(\frac{1}{3}M\dot{y} - \frac{M}{2l}\dot{z}y \right) = -\frac{M}{2l}\dot{z}\dot{y} + \frac{Mg}{2l}y ,$$

which is the same equation twice, so that it is enough to

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look at the equation for $x(t)$, namely

$$\ddot{x} = \frac{3}{2\ell}(g+\ddot{z})\,x = \frac{3g}{2\ell}(1\pm\lambda)\,x.$$

(b) We need a restoring force for one half-period, so that $\lambda>1$ is a first condition. Then

$$\ddot{x} = \omega_+^2 x \quad \text{for half a period}$$

and $\ddot{x} = -\omega_-^2 x$, for the other half period

with $\omega_\pm^2 = \frac{3g}{2\ell}(\lambda\pm1)$.

As in Exercise 36, the mappings

$$\begin{pmatrix} x(t) \\ T\dot{x}(t) \end{pmatrix} \longrightarrow \begin{pmatrix} x(t+\frac{1}{2}T) \\ T\dot{x}(t+\frac{1}{2}T) \end{pmatrix} \longrightarrow \begin{pmatrix} x(t+T) \\ T\dot{x}(t+T) \end{pmatrix}$$

are accomplished by the 2×2 matrices

$$M_+ = \begin{pmatrix} \cosh(\omega_+T/2) & \frac{1}{\omega_+T}\sinh(\omega_+T/2) \\ \omega_+T\sinh(\omega_+T/2) & \cosh(\omega_+T/2) \end{pmatrix}$$

and

$$M_- = \begin{pmatrix} \cos(\omega_-T/2) & \frac{1}{\omega_-T}\sin(\omega_-T/2) \\ -\omega_-T\sin(\omega_-T/2) & \cos(\omega_-T/2) \end{pmatrix}.$$

The rod stays upright if the eigenvalues of M_+M_- (or M_-M_+) do not have absolute values that exceed unity. Upon denoting these eigenvalues by μ_1 and μ_2 we have

$$\mu_1\mu_2 = \det\{M_+M_-\} = \det\{M_+\}\det\{M_-\} = 1$$

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and

$$\mu_1 + \mu_2 = \mathrm{tr}\{M_+ M_-\} = 2\cosh\left(\frac{\omega_+ T}{2}\right)\cos\left(\frac{\omega_- T}{2}\right).$$

So (recall Exercise 36) we have two real eigenvalues $\mu_1 = \pm e^{\alpha}$, $\mu_2 = \pm e^{-\alpha}$ if

$$\cosh\alpha = \left|\cosh\left(\frac{\omega_+ T}{2}\right)\cos\left(\frac{\omega_- T}{2}\right)\right| > 1,$$

and one of these eigenvalues is too big, or we have a pair of complex phase factors, $\mu_1 = e^{i\alpha}$, $\mu_2 = e^{-i\alpha}$ if

$$|\cos\alpha| = \left|\cosh\left(\frac{\omega_+ T}{2}\right)\cos\left(\frac{\omega_- T}{2}\right)\right| \le 1.$$

The condition to be met by λ and T is, therefore,

$$-1 \le \cosh\left(\frac{\omega_+ T}{2}\right)\cos\left(\frac{\omega_- T}{2}\right) \le 1$$

with $\omega_\pm = \sqrt{\frac{3g}{2\ell}(\lambda \pm 1)}$, in addition to $\lambda > 1$.