

Write answers on this side of the paper only.

Do not write on either margin

1] The curl of \vec{F} is $\nabla \times \vec{F} = \nabla \times (m\omega^2 \vec{r} - m\vec{\omega} \vec{\omega} \cdot \vec{r}) = m\vec{\omega} \times \vec{\omega} = 0$,
 so, yes, \vec{F} is conservative, and since \vec{F} is linear in \vec{r} , the
 potential energy is $-\frac{1}{2} \vec{r} \cdot \vec{F} = \frac{m}{2} \vec{r} \cdot [\vec{\omega} \times (\vec{\omega} \times \vec{r})]$
 $= -\frac{m}{2} (\vec{\omega} \times \vec{r})^2$.

2] (a) The second derivative of the potential energy is

$$V''(x) = -\frac{d}{dx} F(x) = 2ax,$$

so that $V''(\pm x_0) = \pm 2ax_0$. It follows that there is
 a stable equilibrium at $x = x_0$ and an unstable
 equilibrium at $x = -x_0$.

(b) For $x \cong x_0$ we have $m\left(\frac{d}{dt}\right)^2(x-x_0) = F(x_0 + (x-x_0))$
 $\cong F'(x_0)(x-x_0)$

or

$$m\left(\frac{d}{dt}\right)^2(x-x_0) \cong -2ax_0(x-x_0) = -m\omega^2(x-x_0)$$

with $\omega = \sqrt{2ax_0/m}$ so that the period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{2ax_0}}$$

(c) The period will be longer because it is very long
 when the energy is close to that required to get to
 the unstable equilibrium point at $x = -x_0$.

3] (a) From

$$L = \frac{m}{2} \vec{v}^2 - V(\vec{r}) + \vec{v} \cdot \vec{\nabla} u(\vec{r})$$

we have

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \vec{v}} - \frac{\partial L}{\partial \vec{r}} = \frac{d}{dt} [m\vec{v} + \vec{\nabla} u(\vec{r})] - [-\vec{\nabla} V(\vec{r}) + \vec{v} \cdot \vec{\nabla} \vec{\nabla} u(\vec{r})]$$

$$= m \frac{d}{dt} \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{\nabla} u(\vec{r}) + \vec{\nabla} V(\vec{r}) - \vec{v} \cdot \vec{\nabla} \vec{\nabla} u(\vec{r})$$

or

$$m \frac{d}{dt} \vec{v} = -\vec{\nabla} V(\vec{r}),$$

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which does not contain any trace of $u(\vec{r})$.

(b) The Hamilton function is, for $\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \nabla u(\vec{r})$,

$$H = \left(\vec{v} \cdot \frac{\partial L}{\partial \vec{v}} - L \right) \Big|_{\vec{v} = [\vec{p} - \nabla u(\vec{r})]/m}$$

$$= \frac{1}{2m} [\vec{p} - \nabla u(\vec{r})]^2 + V(\vec{r}).$$

It depends on the choice of $u(\vec{r})$ because ∇u enters the relation between \vec{p} and \vec{v} .

[4] The body is composed of a homogeneous ball of radius R and an ellipsoid with half-axes $2R, 2R, R$, both having mass density ρ_0 . The total mass is, therefore,

$$M = \frac{4\pi}{3} \rho_0 R^3 + \frac{4\pi}{3} \rho_0 4R^3 = \frac{4\pi}{3} \rho_0 5R^3,$$

so that

$$\rho_0 = \frac{3}{20\pi} \frac{M}{R^3}.$$

(a) In $\vec{I} = \int (d\vec{r}) \rho(\vec{r}) (r^2 \vec{1} - \vec{r}\vec{r})$, we have

$$\int (d\vec{r}) \rho(\vec{r}) \vec{r}\vec{r} = \rho_0 \int_{(\text{ball})} (d\vec{r}) \vec{r}\vec{r} + \rho_0 \int_{(\text{ellipsoid})} (d\vec{r}) \vec{r}\vec{r} = \vec{B} + \vec{E}$$

with

$$\vec{B} = \rho_0 \int_{r < R} (d\vec{r}) \vec{r}\vec{r} = \rho_0 \frac{1}{3} \vec{1} \int_{r < R} (d\vec{r}) r^2 = \rho_0 \frac{4\pi}{3} \frac{R^5}{5} \vec{1}$$

and (recall Exercise 38)

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$$\vec{E} = \rho_0 \int (d\vec{r}') \frac{\vec{r}}{r^3} \gamma \left(1 - \frac{x^2+y^2}{4R^2} - \frac{z^2}{R^2} \right)$$

$$\begin{cases} x=2x' \\ y=2y' \\ z=z' \end{cases}$$

$$\Rightarrow \rho_0 \int_{r' < R} (d\vec{r}') (4x'^2 \vec{e}_x \vec{e}_x + 4y'^2 \vec{e}_y \vec{e}_y + z'^2 \vec{e}_z \vec{e}_z)$$

(terms like $xy \vec{e}_x \vec{e}_y$ are odd and do not contribute)

$$= \rho_0 \frac{4}{3} \int_{r' < R} (d\vec{r}') r'^2 (4\vec{e}_x \vec{e}_x + 4\vec{e}_y \vec{e}_y + \vec{e}_z \vec{e}_z)$$

$$= \rho_0 \frac{4\pi}{3} \frac{4R^5}{5} (4\vec{I} - 3\vec{e}_z \vec{e}_z)$$

Together,

$$\vec{B} + \vec{E} = \frac{4\pi}{3} \rho_0 \frac{R^5}{5} (17\vec{I} - 12\vec{e}_z \vec{e}_z) = \frac{MR^2}{25} (17\vec{I} - 12\vec{e}_z \vec{e}_z)$$

so that

$$\vec{I} = \vec{I} \text{tr}\{\vec{B} + \vec{E}\} - (\vec{B} + \vec{E})$$

$$= \frac{MR^2}{25} [29\vec{I} - (17\vec{I} - 12\vec{e}_z \vec{e}_z)] = \frac{MR^2}{25} (22\vec{I} + 12\vec{e}_z \vec{e}_z)$$

(b) The angular momentum is

$$\vec{L} = \vec{I} \cdot \vec{\omega} = \frac{MR^2}{25} [22\vec{\omega} + 12\vec{e}_z \vec{e}_z \cdot \vec{\omega}]$$

$$= \frac{MR^2}{25} \omega (22\vec{e}_x \sin\theta + 34\vec{e}_z \cos\theta)$$

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