1

(a) On the way up ($0 \le t \le T$), the speed $v(t)$ changes in accordance with

$$\dot v = -g - g\frac{v^2}{v_\infty^2} \quad \text{or} \quad \frac{\dot v/v_\infty}{1+(v/v_\infty)^2} = -\frac{g}{v_\infty}$$

so that

$$\tan^{-1}\left(\frac{v(t)}{v_\infty}\right) + \frac{gt}{v_\infty} = \text{constant}$$

$$\text{or} \quad \frac{v(t)}{v_\infty} = \tan\left(\frac{g}{v_\infty}(T-t)\right) = \frac{v_\infty}{2g}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\cos\left(\frac{g}{v_\infty}(T-t)\right)^2\right)$$

$$\text{with} \quad \tan\left(\frac{gT}{v_\infty}\right) = \frac{v_0}{v_\infty}\,.$$

It follows that the height reached is

$$h = \int_0^T \mathrm{d}t\; v(t) = -\frac{v_\infty^2}{2g}\log\left(\cos\left(\frac{gT}{v_\infty}\right)^2\right) = \frac{v_\infty^2}{2g}\log\left(1+\tan\left(\frac{gT}{v_\infty}\right)^2\right)$$

or, finally,

$$h = \frac{v_\infty^2}{2g}\log\frac{v_\infty^2+v_0^2}{v_\infty^2}\,.$$

(b) On the way down ($t \ge T$), the speed $v(t)$ changes in accordance with

$$\dot v = g - g\frac{v^2}{v_\infty^2} \quad \text{or} \quad \frac{\dot v/v_\infty}{1-(v/v_\infty)^2} = \frac{g}{v_\infty}$$

so that

$$\tanh^{-1}\left(\frac{v(t)}{v_\infty}\right) - \frac{gt}{v_\infty} = \text{constant}$$

$$\text{or} \quad \frac{v(t)}{v_\infty} = \tanh\left(\frac{g}{v_\infty}(t-T)\right) = \frac{v_\infty}{2g}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\cosh\left(\frac{g}{v_\infty}(t-T)\right)^2\right)\,.$$

It follows that the height above ground at time t is

$$\begin{aligned} h - \int_T^t \mathrm{d}t'\; v(t') &= h - \frac{v_\infty^2}{2g}\log\left(\cosh\left(\frac{g}{v_\infty}(t-T)\right)^2\right) \\ &= h + \frac{v_\infty^2}{2g}\log\left(1-\tanh\left(\frac{g}{v_\infty}(t-T)\right)^2\right) \\ &= h - \frac{v_\infty^2}{2g}\log\frac{v_\infty^2}{v_\infty^2 - v(t)^2}\,. \end{aligned}$$

Therefore, v_1 is given by

$$h = \frac{v_\infty^2}{2g} \log \frac{v_\infty^2}{v_\infty^2 - v_1^2} .$$

It follows that

$$\frac{v_\infty^2 + v_0^2}{v_\infty^2} = \frac{v_\infty^2}{v_\infty^2 - v_1^2} \quad \text{or} \quad v_1 = \frac{v_0 v_\infty}{\sqrt{v_\infty^2 + v_0^2}} .$$

2

(a) At equilibrium, the masses are distance a apart and each at distance a from the adjacent wall, and all springs are relaxed; we choose x_1 and x_2 as the displacements from equilibrium to the right. Then

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k_1(x_1^2 + x_2^2) - \frac{1}{2}k_2(x_2 - x_1)^2 = \frac{1}{2}\dot{X}^{\mathrm{T}} M \dot{X} - \frac{1}{2}X^{\mathrm{T}} K X$$

with

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{pmatrix} .$$

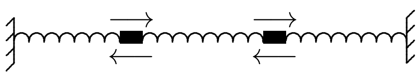
(b) Since $X^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $X^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are common eigencolumns of M and K, they specify the normal modes, and we find the characteristic frequencies ω_1 and ω_2 from

$$0 = (\omega_1^2 M - K)X^{(1)} = (m\omega_1^2 - k_1)X^{(1)} \quad \text{so that} \quad \omega_1 = \sqrt{\frac{k_1}{m}} ,$$

$$0 = (\omega_2^2 M - K)X^{(2)} = (m\omega_2^2 - k_1 - 2k_2)X^{(2)} \quad \text{so that} \quad \omega_2 = \sqrt{\frac{k_1 + 2k_2}{m}} .$$

Since $\omega_1 < \omega_2$, the first mode is the slow one, and the second mode is the fast one.

(c) The first normal mode is just center-of-mass motion, where the distance between the masses is a at all times and the inner spring is always relaxed:



The second normal mode is a breathing mode, where the center-of-mass is at rest and the two masses move with opposite velocities:

(d) The Hamilton function is

$$H = \frac{1}{2}P^{\mathrm{T}}M^{-1}P + \frac{1}{2}X^{\mathrm{T}}KX = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}k_1(x_1^2 + x_2^2) + \frac{1}{2}k_2(x_2 - x_1)^2 \,.$$

3

(a) The center-of-mass of the two-body system is at

$$\boldsymbol{R} = \frac{1}{M_1 + M_2}(M_1\boldsymbol{R}_1 + M_2\boldsymbol{R}_2)\,,$$

and the positions of bodies 1 and 2 relative to the center-of-mass are

$$\boldsymbol{R}_1 - \boldsymbol{R} = \frac{M_2}{M_1 + M_2}(\boldsymbol{R}_1 - \boldsymbol{R}_2) \quad \text{and} \quad \boldsymbol{R}_2 - \boldsymbol{R} = \frac{M_1}{M_1 + M_2}(\boldsymbol{R}_2 - \boldsymbol{R}_1)\,.$$

Upon applying Steiner' theorem twice, we get

$$\begin{aligned}\mathsf{I} &= \mathsf{I}_1 + M_1\big[(\boldsymbol{R}_1 - \boldsymbol{R})^2\boldsymbol{1} - (\boldsymbol{R}_1 - \boldsymbol{R})\,(\boldsymbol{R}_1 - \boldsymbol{R})\big] \\ &\quad + \mathsf{I}_2 + M_2\big[(\boldsymbol{R}_2 - \boldsymbol{R})^2\boldsymbol{1} - (\boldsymbol{R}_2 - \boldsymbol{R})\,(\boldsymbol{R}_2 - \boldsymbol{R})\big] \\ &= \mathsf{I}_1 + \mathsf{I}_2 + \frac{M_1M_2}{M_1 + M_2}\big[(\boldsymbol{R}_1 - \boldsymbol{R}_2)^2\boldsymbol{1} - (\boldsymbol{R}_1 - \boldsymbol{R}_2)\,(\boldsymbol{R}_1 - \boldsymbol{R}_2)\big]\,.\end{aligned}$$

(b) We denote the position vectors of the four point masses by $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, and $\boldsymbol{r}_4$. Their cartesian coordinates could be

$$\begin{pmatrix}\boldsymbol{r}_1 & \boldsymbol{r}_2 & \boldsymbol{r}_3 & \boldsymbol{r}_4\end{pmatrix} \,\widehat{=}\, \frac{a}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix},$$

for example, and we have $\boldsymbol{r}_j \cdot \boldsymbol{r}_k = a^2\delta_{jk} - \frac{1}{4}a^2$ as well as $\sum_{j=1}^{4}\boldsymbol{r}_j\,\boldsymbol{r}_j = a^2\boldsymbol{1}$ for their dot products and the sum of their dyadic squares. Accordingly, the inertia dyadic is

$$\mathsf{I}_{4\text{ masses}} = \sum_{j=1}^{4} m(r_j^2\boldsymbol{1} - \boldsymbol{r}_j\,\boldsymbol{r}_j) = 2ma^2\boldsymbol{1}\,.$$

(c) We recognize the situation of part (a) for the three-mass system as body 1 ($M_1 = 3m$, $\boldsymbol{R}_1 = -\frac{1}{3}\boldsymbol{r}_4$, and $\mathsf{I}_1 = \mathsf{I}_{3\text{ masses}}$) and the fourth mass as body 2 ($M_2 = m$, $\boldsymbol{R}_2 = \boldsymbol{r}_4$, and $\mathsf{I}_2 = \boldsymbol{0}$), and the two-body system is the four-mass system. With $a^2 = \frac{4}{3}r_4^2$, we then have

$$\mathsf{I}_{4\text{ masses}} = \frac{8}{3}mr_4^2\mathbf{1} = \mathsf{I}_{3\text{ masses}} + \mathbf{0} + \frac{3m^2}{4m}\left[\left(\frac{4}{3}\boldsymbol{r}_4\right)^2\mathbf{1} - \frac{4}{3}\boldsymbol{r}_4\,\frac{4}{3}\boldsymbol{r}_4\right]$$

and find

$$\mathsf{I}_{3\text{ masses}} = \frac{4}{3}m\bigl(r_4^2\mathbf{1} + \boldsymbol{r}_4\,\boldsymbol{r}_4\bigr)\,.$$

4

(a) In the laboratory frame we have $m\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla}V = -m\omega_0^2\bigl[\boldsymbol{r} - 3\boldsymbol{n}\,\boldsymbol{n}\cdot\boldsymbol{r}\bigr]$.

(b) We introduce coordinates in the rotating frame by writing

$$\boldsymbol{r} = x\boldsymbol{n} + y\boldsymbol{e}_z\times\boldsymbol{n} + z\boldsymbol{e}_z \mathrel{\widehat{=}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

With

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{n} = \Omega\boldsymbol{e}_z\times\boldsymbol{n} \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_z\times\boldsymbol{n} = -\Omega\boldsymbol{n}$$

we then have

$$\dot{\boldsymbol{r}} \mathrel{\widehat{=}} \begin{pmatrix} \dot{x} - \Omega y \\ \dot{y} + \Omega x \\ \dot{z} \end{pmatrix},\quad \ddot{\boldsymbol{r}} \mathrel{\widehat{=}} \begin{pmatrix} \ddot{x} - 2\Omega\dot{y} - \Omega^2 x \\ \ddot{y} + 2\Omega\dot{x} - \Omega^2 y \\ \ddot{z} \end{pmatrix} \quad\text{and}\quad \boldsymbol{r} - 3\boldsymbol{n}\,\boldsymbol{n}\cdot\boldsymbol{r} \mathrel{\widehat{=}} \begin{pmatrix} -2x \\ y \\ z \end{pmatrix}.$$

Together they gives us the equation of motion

$$\begin{pmatrix} \ddot{x} - 2\Omega\dot{y} - (2\omega_0^2 + \Omega^2)x \\ \ddot{y} + 2\Omega\dot{x} + (\omega_0^2 - \Omega^2)y \\ \ddot{z} + \omega_0^2 z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) The z motion is harmonic all by itself and thus stable, irrespective of the value of Ω. For the coupled xy motion, we make the exponential ansatz $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{e}^{\mathrm{i}\omega t}\begin{pmatrix} a \\ b \end{pmatrix}$, where ω must be real to ensure that the point mass stays near $\boldsymbol{r} = 0$. The ansatz works if

$$\begin{pmatrix} \Omega^2 + 2\omega_0^2 + \omega^2 & 2\mathrm{i}\Omega\omega \\ -2\mathrm{i}\Omega\omega & \Omega^2 - \omega_0^2 + \omega^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0\,,$$

which requires that the determinant of the 2×2 matrix vanishes. The possible ω^2 values are, therefore, solutions of

$$(\Omega^2 + 2\omega_0^2 + \omega^2)(\Omega^2 - \omega_0^2 + \omega^2) - 4\Omega^2\omega^2 = 0$$

or $$\left(\omega^2 + \frac{1}{2}\omega_0^2 - \Omega^2\right)^2 - \frac{1}{4}\omega_0^2(9\omega_0^2 - 8\Omega^2) = 0\,,$$

which are two versions of the same second-degree polynomial in ω^2. This polynomial has two positive roots if (i) its value is positive for $\omega^2 = 0$; (ii) its minimum is located at a positive ω^2 value; and (iii) the minimum is negative. Accordingly, we need

$$\text{(i)}\quad (\Omega^2 + 2\omega_0^2)(\Omega^2 - \omega_0^2) > 0\,,$$
$$\text{(ii)}\quad \Omega^2 - \frac{1}{2}\omega_0^2 > 0\,,$$
$$\text{(iii)}\quad \frac{1}{4}\omega_0^2(9\omega_0^2 - 8\Omega^2) > 0\,.$$

It follows that the point mass stays near $\boldsymbol{r} = 0$ if

$$\omega_0^2 < \Omega^2 < \frac{9}{8}\omega_0^2\,.$$