1 We need to eliminate the fugacity z from

$$\rho\lambda^3 = g_{\frac{3}{2}}(z) = z + 2^{-3/2}z^2 + \cdots \quad \text{and} \quad \beta P\lambda^3 = g_{\frac{5}{2}}(z) = z + 2^{-5/2}z^2 + \cdots$$

for $0 < z \ll 1$. This gives first $z \cong \rho\lambda^3 - 2^{-3/2}(\rho\lambda^3)^2$, and then $\beta P\lambda^3 \cong \rho\lambda^3 - 2^{-3/2}(\rho\lambda^3)^2 + 2^{-5/2}(\rho\lambda^3)^2$ or, finally,

$$\beta P \cong \rho - 2^{-5/2}\rho^2\lambda^3\,.$$

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2 The canonical partition function is

$$Q(\beta,V,N) = \frac{1}{N!}\left[\frac{V}{(2\pi\hbar)^3}\int(\mathrm{d}\boldsymbol{p})\mathrm{e}^{-\beta c|\boldsymbol{p}|}\right]^N = \frac{1}{N!}\left[\frac{V}{(2\pi\hbar)^3}\frac{8\pi}{(\beta c)^3}\right]^N,$$

and the free energy is

$$F(\beta,V,N) = -\frac{1}{\beta}\log Q = -\frac{N}{\beta} + \frac{N}{\beta}\log\frac{\pi^2(\beta\hbar c)^3}{V/N}\,.$$

This yields

$$P = -\frac{\partial F}{\partial V} = \frac{N}{\beta V} \quad \text{and} \quad U = F + TS = F + \beta\frac{\partial F}{\partial \beta} = \frac{\partial(\beta F)}{\partial\beta} = \frac{3N}{\beta}\,,$$

so that $u = \dfrac{U}{V} = 3P$. This is as expected, since we have a state density $\propto p^2\mathrm{d}p \propto \varepsilon^2\mathrm{d}\varepsilon$ with $\kappa = 3$ in (3.9.3) and (3.9.6).

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3 We consider $E_{\mathrm{TF}}[\rho]-E_{\mathrm{TF}}[\rho_{\mathrm{TF}}] = \Big(E_{\mathrm{TF}}[\rho_{\mathrm{TF}}+x(\rho-\rho_{\mathrm{TF}})]-E_{\mathrm{TF}}[\rho_{\mathrm{TF}}]\Big)_{x=1} \equiv f(x)\Big|_{x=1}$ and note that $f(1) = f(0)+f'(0)+\frac{1}{2}f''(y)$ with $0 \le y \le 1$. Here, $f(0) = 0$ by construction and $f'(0) = 0$ since $E_{\mathrm{TF}}[\rho]$ is stationary at ρ_{TF}, and we need to verify that $f''(y) \ge 0$. With $\Delta(\boldsymbol{r}) = \rho(\boldsymbol{r}) - \rho_{\mathrm{TF}}(\boldsymbol{r})$, we have

$$\begin{aligned} f''(y) = {} & \frac{\hbar^2}{10\pi^2 m}\int(\mathrm{d}\boldsymbol{r})\,(3\pi^2)^{5/3}\,\frac{5}{3}\,\frac{2}{3}\,\Delta(\boldsymbol{r})^2\left[\rho_{\mathrm{TF}}(\boldsymbol{r}) + y\Delta(\boldsymbol{r})\right]^{-1/3} \\ & + \frac{e^2}{2}\int(\mathrm{d}\boldsymbol{r})(\mathrm{d}\boldsymbol{r}')\,\frac{\Delta(\boldsymbol{r})\,\Delta(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,, \end{aligned}$$

where both terms are positive. It follows that $f''(y) \ge 0$.

4

(a) Proceeding from

$$\begin{aligned} E_k &= \sum_j \left\{ \begin{array}{l} +J \ \text{ if } s_j s_{j+1} = -1 \\ -J_+ \text{ if } s_j = s_{j+1} = +1 \\ -J_- \text{ if } s_j = s_{j+1} = -1 \end{array} \right\} \\ &= \sum_j \frac{1}{4}\left[(2J - J_+ - J_-) + (J_- - J_+)(s_j + s_{j+1}) - (2J + J_+ + J_-)s_j s_{j+1}\right] \\ &= \frac{1}{4}N(2J - J_+ - J_-) + \frac{1}{2}(J_- - J_+)\sum_j s_j - \frac{1}{4}(2J + J_+ + J_-)\sum_j s_j s_{j+1} \end{aligned}$$

we identify

$$\mathcal{E} = \frac{1}{4}(2J - J_+ - J_-)\,, \quad E_0' = J_- - J_+\,, \quad J' = \frac{1}{4}(2J + J_+ + J_-)\,.$$

(b) The partition function is

$$Q(\beta J, \beta J_+, \beta J_-, N) = \mathrm{e}^{-\beta N \mathcal{E}} \lambda_+(\beta E_0', \beta J')^N$$

with $\lambda_+(\beta E_0, \beta J)$ from (4.2.36) in (4.2.38), where the λ_-^N term is negligibly small.

(c) Here we have $\mathcal{E} = 0$, $E_0' = -2\epsilon$, and $J' = J$. In $F = N\mathcal{E} - \dfrac{N}{\beta}\log \lambda_+(\beta E_0', \beta J')$, we need $\lambda_+(\beta E_0, \beta J)$ to first-order in βE_0, which is $\lambda_+(0, \beta J) = 2\cosh(\beta J)$ as there are no first-order terms. Accordingly, we obtain

$$F = -\frac{N}{\beta}\log\Big(2\cosh(\beta J)\Big) + \cdots\,,$$

where the ellipsis stands for terms of second and higher order in ϵ.