Problem 1 (**25** marks)
For an ideal gas of bosons at low density or high temperature, what is the leading correction to $\beta P = \rho$?

Problem 2 (**25** marks)
Imagine an ideal classical gas of N "luxons" — that is: massless, conserved particles with kinetic energy $c|\boldsymbol{p}|$, where c is the speed of light — confined to volume V. Determine the energy density u and the pressure P as functions of the temperature T. Comment on what you find.

Problem 3 (**20** marks)
Thomas–Fermi model of the atom (Section 4.1 in the lecture notes). We know that the TF functional $E_{\mathrm{TF}}[\rho]$ in (4.1.5) is stationary at the TF density ρ_{TF}. Now verify that the extremum is a minimum, that is: show that $E_{\mathrm{TF}}[\rho] > E_{\mathrm{TF}}[\rho_{\mathrm{TF}}]$ for all other permissible ρ.

Problem 4 (**30**=10+10+10 marks)
Consider a one-dimensional Ising chain (or ring) with N links between next neighbors. There is no on-site energy, and the next-neighbor interaction energy for sites j and $j+1$ is

$$\left\{ \begin{array}{l} +J \text{ if } s_j s_{j+1} = -1 \\ -J_+ \text{ if } s_j = s_{j+1} = +1 \\ -J_- \text{ if } s_j = s_{j+1} = -1 \end{array} \right\} = J \frac{1 - s_j s_{j+1}}{2} - J_+ \frac{(1+s_j)(1+s_{j+1})}{4} - J_- \frac{(1-s_j)(1-s_{j+1})}{4} \,.$$

We have the standard Ising model for $J_+ = J_- = J$ and a modified Ising model otherwise.

(a) Show that such a modified Ising model is equivalent to a standard Ising model with a certain on-site energy E_0' and a certain interaction energy J' plus an energy off-set $\mathcal{E}$, in the sense that the energy of the kth microstate can be written as

$$E_k = N\mathcal{E} + \frac{1}{2} E_0' \sum_j s_j - J' \sum_j s_j s_{j+1} \,.$$

State $\mathcal{E}$, E_0', and J' in terms of J, J_+, and J_-.

(b) Find the canonical partition function of this modified Ising model.

(c) When $J_\pm = J \pm \epsilon$, what is the free energy to first order in ϵ?