1 Quite generally, we have

$$\beta PV = \log\bigl(Z(\beta,V,z)\bigr) \quad \text{and} \quad N = z\left(\frac{\partial \log Z}{\partial z}\right)_{\beta,V}$$

from lecture and also

$$\frac{S}{k_\mathrm{B}} = -\beta^2\left(\frac{\partial(\beta^{-1}\log Z)}{\partial\beta}\right)_{V,z} - N\log z\,.$$

from Exercise 33. For $\log\bigl(Z(\beta,V,z)\bigr) = \dfrac{V}{\lambda^3}h(z)$ with $\lambda = \hbar\sqrt{2\pi\beta/m} \propto \beta^{\frac{1}{2}}$, these give

$$\beta PV = \frac{V}{\lambda^3}h(z) \quad \text{and} \quad N = \frac{V}{\lambda^3}\,zh'(z)$$

$$\text{and} \qquad \frac{S}{k_\mathrm{B}} = \frac{5}{2}\frac{V}{\lambda^3}h(z) - N\log z \quad \text{or} \quad \frac{S}{k_\mathrm{B}N} = \frac{5}{2}\frac{h(z)}{zh'(z)} - \log z\,.$$

It follows that z is constant when S and N are, that is: when we consider isentropic changes. Then, all terms on the right-hand side of

$$\begin{aligned} P^3V^5 &= \left(\frac{h(z)}{\beta\lambda^3}\right)^3\left(\frac{N\lambda^3}{zh'(z)}\right)^5 = h(z)^3\left(\frac{N}{zh'(z)}\right)^5\left(\frac{\lambda^2}{\beta}\right)^3 \\ &= h(z)^3\left(\frac{N}{zh'(z)}\right)^5\left(\frac{2\pi\hbar^2}{m}\right)^3 \end{aligned}$$

are constant as well, so that $P^3V^5 = \text{constant}$ holds irrespective of the particular $h(z)$.

Alternatively, we can use

$$\langle E\rangle = -\left(\frac{\partial\log Z}{\partial\beta}\right)_{V,z} = \frac{3}{2\beta}\log Z = \frac{3}{2}PV\,,$$

and identify $\langle E\rangle$ with the internal energy U. For isentropic changes ($\mathrm{d}S = 0$ and $\mathrm{d}N = 0$), we have $(\mathrm{d}U)_{S,N} = -P(\mathrm{d}V)_{S,N}$, so that

$$-P = \left(\frac{\partial\langle E\rangle}{\partial V}\right)_{S,N} = \frac{3}{2}\left(\frac{\partial(PV)}{\partial V}\right)_{S,N} = \frac{3}{2}P + \frac{3}{2}V\left(\frac{\partial P}{\partial V}\right)_{S,N}$$

or

$$\left(\frac{\partial P}{\partial V}\right)_{S,N} = -\frac{5}{3}\frac{P}{V}\,.$$

This implies that $P \propto V^{-5/3}$ or $P^3V^5 = \text{constant}$ for istentropic changes.

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2 Since

$$Q(\beta E_0, \beta J, N) = \sum_k \mathrm{e}^{-\frac{1}{2}\beta E_0\sum_j s_j + \beta J\sum_j s_j s_{j+1}} = \lambda_+^N$$

with $\lambda_\pm = \lambda_\pm(\beta E_0, \beta J)$ given in (4.2.36) in the lecture notes, we have

$$\sum_j \langle s_j \rangle = -2\left(\frac{\partial \log Q}{\partial(\beta E_0)}\right)_{\beta J, N} = -2N\left(\frac{\partial \log \lambda_+}{\partial(\beta E_0)}\right)_{\beta J}$$

$$\text{and} \quad \sum_j \langle s_j s_{j+1} \rangle = \left(\frac{\partial \log Q}{\partial(\beta J)}\right)_{\beta E_0, N} = N\left(\frac{\partial \log \lambda_+}{\partial(\beta J)}\right)_{\beta E_0} .$$

We also have

$$\sum_j \langle s_j \rangle = \langle N_+ \rangle - \langle N_- \rangle \quad \text{with} \quad \langle N_+ \rangle + \langle N_- \rangle = N$$

$$\text{and} \quad \sum_j \langle s_j s_{j+1} \rangle = \langle N_+^{(\mathrm{nn})} \rangle - \langle N_-^{(\mathrm{nn})} \rangle \quad \text{with} \quad \langle N_+^{(\mathrm{nn})} \rangle + \langle N_-^{(\mathrm{nn})} \rangle = N .$$

Accordingly, we obtain

$$\langle N_\pm \rangle = \frac{1}{2}\left(N \pm \sum_j \langle s_j \rangle\right) = \frac{N}{2}\left[1 \mp 2\left(\frac{\partial \log \lambda_+}{\partial(\beta E_0)}\right)_{\beta J}\right]$$

$$\text{and} \quad \langle N_\pm^{(\mathrm{nn})} \rangle = \frac{1}{2}\left(N \pm \sum_j \langle s_j s_{j+1} \rangle\right) = \frac{N}{2}\left[1 \pm \left(\frac{\partial \log \lambda_+}{\partial(\beta J)}\right)_{\beta E_0}\right] ,$$

where

$$\left(\frac{\partial \log \lambda_+}{\partial(\beta J)}\right)_{\beta E_0} = \frac{2\lambda_+ \mathrm{e}^{\beta J} \cosh(\frac{1}{2}\beta E_0) - 4\cosh(2\beta J)}{(\lambda_+ - \lambda_-)\lambda_+}$$

and, from (4.2.50),

$$\left(\frac{\partial \log \lambda_+}{\partial(\beta E_0)}\right)_{\beta J} = \frac{\mathrm{e}^{\beta J} \sinh(\frac{1}{2}\beta E_0)}{\lambda_+ - \lambda_-} .$$

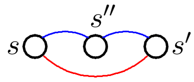
3

(a) Here we have a standard Ising chain, for which

$$F(K, 0, N) = -\frac{N}{\beta} \log\bigl(2\cosh(K)\bigr) .$$

(b) Here we have $\frac{1}{2}N$ isolated sites, and a chain with $\frac{1}{2}N$ sites and next-neighbor interaction energy J', so that

$$\begin{aligned} F(0, K', N) &= F(0, 0, \tfrac{1}{2}N) + F(K', 0, \tfrac{1}{2}N) \\ &= -\frac{N}{2\beta}\Bigl(\log(2) + \log\bigl(2\cosh(K')\bigr)\Bigr) \\ &= -\frac{N}{2\beta} \log\bigl(4\cosh(K')\bigr) . \end{aligned}$$

(c) We look at one element of three sites with two J links and one J' link: 

which contributes a factor

$$\sum_{s''=\pm 1} \mathrm{e}^{Kss'' + Ks''s' + K'ss'} = 2\cosh\big((s+s')K\big)\mathrm{e}^{K'ss'} = M_{ss'}\,,$$

with the 2×2 matrix

$$M = 2\begin{pmatrix} \cosh(2K)\mathrm{e}^{K'} & \mathrm{e}^{-K'} \\ \mathrm{e}^{-K'} & \cosh(2K)\mathrm{e}^{K'} \end{pmatrix},$$

and there are $\frac{1}{2}N$ (or $\frac{1}{2}N - 1$) such matrices in

$$Q(K,K',N) = \begin{pmatrix} 1 & 1 \end{pmatrix} M^{\frac{1}{2}N} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since

$$M\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} 2\Big(\cosh(2K)\mathrm{e}^{K'} + \mathrm{e}^{-K'}\Big),$$

we obtain

$$\begin{aligned} F(K,K',N) &= -\frac{N}{2\beta}\log\Big(2\cosh(2K)\mathrm{e}^{K'} + 2\mathrm{e}^{-K'}\Big) \\ &= -\frac{N}{2\beta}\log\Big(4\cosh(K)^2\cosh(K') + 4\sinh(K)^2\sinh(K')\Big). \end{aligned}$$

Special cases are

$$F(K,0,N) = -\frac{N}{2\beta}\log\Big(2\cosh(2K) + 2\Big) = -\frac{N}{\beta}\log\Big(2\cosh(K)\Big)$$

and

$$F(0,K',N) = -\frac{N}{2\beta}\log\Big(2\mathrm{e}^{K'} + 2\mathrm{e}^{-K'}\Big) = -\frac{N}{2\beta}\log\big(4\cosh(K')\big).$$

They agree with the expressions in (a) and (b), as they should.

(d) As in (4.5.1) with (4.5.2), the heat capacity is

$$C = T\frac{\partial^2}{\partial T^2}k_{\mathrm{B}}T\log\bigl(Q(K,K',N)\bigr) = k_{\mathrm{B}}\beta^2\frac{\partial^2}{\partial\beta^2}\log\bigl(Q(K,K',N)\bigr)\,.$$

With

$$\left(\frac{\partial}{\partial\beta}\right)^2 = \left(J\frac{\partial}{\partial K} + J'\frac{\partial}{\partial K'}\right)^2 = J^2\frac{\partial^2}{\partial K^2} + 2JJ'\frac{\partial}{\partial K}\frac{\partial}{\partial K'} + J'^2\frac{\partial^2}{\partial K'^2}$$

this becomes

$$\frac{C}{k_{\mathrm{B}}N} = \left(K^2\frac{\partial^2}{\partial K^2} + 2KK'\frac{\partial}{\partial K}\frac{\partial}{\partial K'} + K'^2\frac{\partial^2}{\partial K'^2}\right)\frac{1}{2}\log\Bigl(\cosh(2K)\mathrm{e}^{K'} + \mathrm{e}^{-K'}\Bigr)\,.$$

For $K' = 0$, we obtain

$$\left.\frac{C}{k_{\mathrm{B}}N}\right|_{K'=0} = K^2\frac{\partial^2}{\partial K^2}\log\bigl(\cosh(K)\bigr) = \frac{K^2}{\cosh(K)^2}\,.$$

For corrections of order K', we use

$$\begin{aligned}\frac{1}{2}\log\Bigl(\cosh(2K)\mathrm{e}^{K'} + \mathrm{e}^{-K'}\Bigr) &= \frac{1}{2}\log\Bigl(4\cosh(K)^2 + 4\sinh(K)^2K'\Bigr) + \cdots\\ &= \log\bigl(2\cosh(K)\bigr) + \frac{1}{2}\log\Bigl(1 + \tanh(K)^2K'\Bigr) + \cdots\\ &= \log\bigl(2\cosh(K)\bigr) + \frac{1}{2}\tanh(K)^2K' + \cdots\,,\end{aligned}$$

where the ellipsis stands for terms of order K'^2 or higher. The first-order correction to the heat capacity is thus given by

$$\begin{aligned}\left.\frac{C}{k_{\mathrm{B}}N}\right|_{\mathrm{1st}} &= \left(K^2\frac{\partial^2}{\partial K^2} + 2KK'\frac{\partial}{\partial K}\frac{\partial}{\partial K'}\right)\frac{1}{2}\tanh(K)^2K'\\ &= K^2K'\left(\frac{3}{\cosh(K)^4} - \frac{2}{\cosh(K)^2}\right) + 2KK'\frac{\sinh(K)}{\cosh(K)^3}\,.\end{aligned}$$