PC1134 Lecture 1

Topic

- Review of differentiation

Objectives

To become familiar with

- ordinary differentiation
- basic techniques of differentiation
- derivatives of common functions

Relevance

- These concepts and techniques are important for the study of partial differentiation.

Function

what is a function?

In mathematics, especially in its applications to physical science, we are often interested in the relations and connections between different numbers or sets of numbers. A *function* is a way of expressing such a connection.

$$y = f(x)$$

x is referred as independent variable, y is the dependent variable.

Examples of function

- *Position* of an moving object can be a function of *time*.

$$x(t) = x_0 + vt + \frac{1}{2}at^2$$

- *Potential* of a point charge is a function of r, distance from the point charge.

$$V(r) = \frac{1}{4\pi\varepsilon}\frac{Q}{r}$$

- The *temperature* of an ideal gas is a *function* of pressure for *constant* volume

$$PV = nRT$$

where n and R are referred as constants or *parameters*

Representing Function

A function can be represented by

- an analytic **equation**

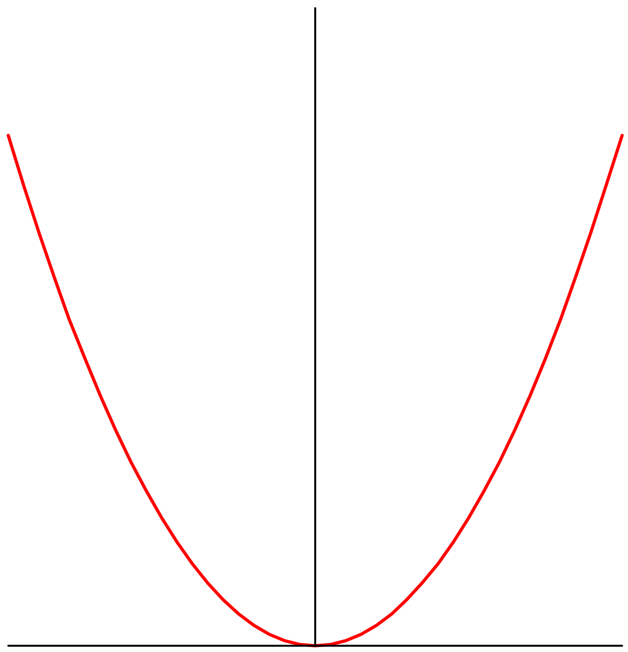
$$V = \frac{1}{2}kx^2$$

- a **table** (for discrete variable)

Time	Temperature (°C)
8:00	25.0
9:00	26.2
10:00	27.5
11:00	28.7
12:00	30.0
13:00	31.4
14:00	32.6
15:00	32.0
16:00	31.3
17:00	30.8

Representing Function (cont.)

- or a **graph**



Derivative

Given $y = f(x)$, a change in x will cause a change in y.

$$x \longrightarrow x + \Delta x$$
$$y \longrightarrow y + \Delta y$$

How is Δy related to Δx?

$$y = f(x)$$

$$y + \Delta y = f(x + \Delta x)$$
$$\Delta y = f(x + \Delta x) - f(x)$$

"Rate" of change

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

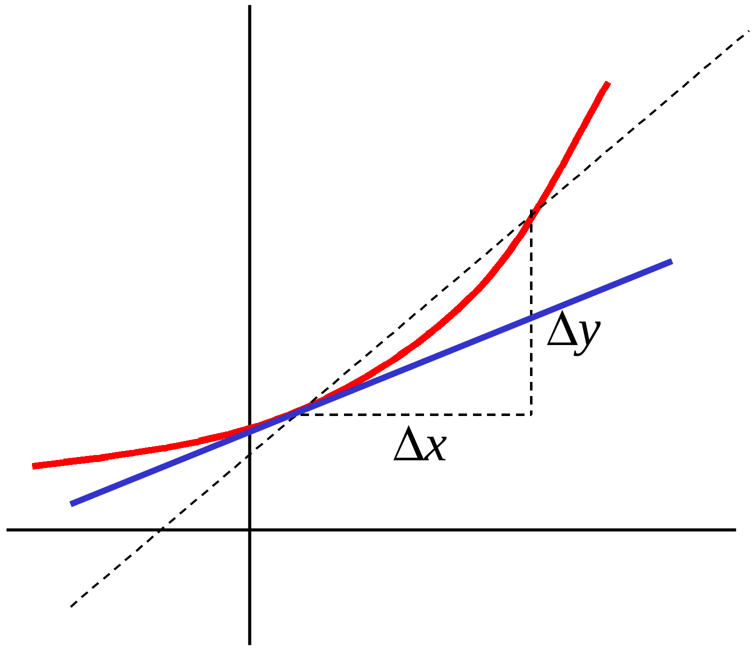
When $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = y'$$

Derivative

$\dfrac{dy}{dx}$ is the rate of change of y with respect to x.

Graphically, the value of $\dfrac{dy}{dx}$ at any particular value of x is equal to the gradient of the tangent to the graph of y against x at that particular value of x.



Higher Derivatives

In general, $\dfrac{dy}{dx}$ is a function of x and can be written as $y'(x)$, $f'(x)$ or $\dfrac{dy}{dx}(x)$.

Let

$$z(x) = f'(x)$$

Since $z(x)$ is a function of x, we can calculate its derivative

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$$

This is called the second derivative of y with respect to x and is written as

$$\frac{d^2y}{dx^2}, \text{ or } y'' \text{ or } f''(x)$$

Higher derivatives can be defined similarly.

Rules

- If
  $$y = f(x) \pm g(x)$$
  then
  $$y' = f' \pm g'$$
- If
  $$y = f(x)g(x)$$
  then
  $$y' = fg' + f'g$$
- If
  $$y = \frac{f(x)}{g(x)}$$
  then
  $$y' = \frac{f'g - fg'}{g^2}$$

Chain Rule

Function of a function:

$$
\begin{aligned}
y &= y(x) \\
x &= x(t)
\end{aligned}
$$

$$\frac{dy}{dt} = ?$$

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta t}$$

$\Longrightarrow$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Inverse Function

$$y = f(x) \Longrightarrow x = g(y)$$

$$\frac{\Delta x}{\Delta y} = \frac{1}{\dfrac{\Delta y}{\Delta x}}$$

$\Longrightarrow$

$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}}$$

Implicit Differentiation

Consider the function defined by the equation

$$x^3 - 3xy + y^3 = 2$$

It cannot be resented explicitly in the form of

$$y = f(x)$$

We can differentiate term by term with respect to x

$$\frac{d}{dx}\left(x^3\right) - \frac{d}{dx}(3xy) + \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(2)$$

$$\Longrightarrow 3x^2 - \left(3x\frac{dy}{dx} + 3y\right) + 3y^2\frac{dy}{dx} = 0$$

Rearranging terms gives

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Function Defined Parametrically

Given

$$\begin{aligned} y &= y(t) \\ x &= x(t) \end{aligned}$$

What is $\dfrac{dy}{dx}$?

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t}\frac{\Delta t}{\Delta x}$$

$\Longrightarrow$

$$\frac{dy}{dx} = \frac{dy}{dt}\Bigg/\frac{dx}{dt}$$

Derivatives of Elementary Functions

Algebraic function

$$y = x^r \qquad \frac{dy}{dx} = rx^{r-1}$$

Trigonometrical function

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} \qquad \frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}$$

Inverse trigonometrical function

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

Derivatives of Elementary Functions

Exponential function

$$y = e^x \qquad \frac{dy}{dx} = e^x = y$$

Logarithmic function

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

Hyperbolic functions

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) \qquad \frac{d}{dx}\sinh x = \cosh x$$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$$

$$\coth x = \frac{\cosh x}{\sinh x} \qquad \frac{d}{dx}\coth x = \frac{1}{\sinh^2 x}$$

Derivatives of Elementary Functions

Inverse hyperbolic functions

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2} = \frac{d}{dx}\coth^{-1}x$$

Example 1

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

Let

$$y = \sin^{-1} z \text{ and } z = 2x\sqrt{1-x^2}$$

then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$$\frac{dy}{dz} = \frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}$$

Treating $2x\sqrt{1-x^2}$ as the product of two function $2x$ and $\sqrt{1-x^2}$

$$\frac{dz}{dx} = \frac{d}{dx}(2x)\cdot\sqrt{1-x^2} + 2x\frac{d}{dx}\sqrt{1-x^2}$$

Example 1

The derivative of $\sqrt{1-x^2}$ can be evaluated by apply the chain rule again. The result is

$$\frac{d}{dx}\sqrt{1-x^2} = -\frac{x}{\sqrt{1-x^2}}$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-z^2}}2\sqrt{1-x^2} + 2x\frac{-x}{\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-4x^2(1-x^2)}}\frac{2(1-2x^2)}{\sqrt{1-x^2}} \\ &= \pm\frac{2}{\sqrt{1-x^2}}\end{aligned}$$

Where does the $\pm$ come from?

Example 2

If $y = x^5 + x$, find $\dfrac{d^2x}{dy^2}$.

$$\frac{dy}{dx} = 5x^4 + 1$$

$$\frac{dx}{dy} = \frac{1}{5x^4 + 1}$$

$$\begin{aligned} \frac{d^2x}{dy^2} &= \frac{d}{dy}\left(\frac{1}{5x^4+1}\right) \\ &= \frac{d}{dx}\left(\frac{1}{5x^4+1}\right)\frac{dx}{dy} \\ &= -\frac{20x^3}{(5x^4+1)^2}\cdot\frac{1}{5x^4+1} \\ &= -\frac{20x^3}{(5x^4+1)^3} \end{aligned}$$

Example 3

$$x = a\cos^3 t, \qquad y = a\sin^3 t$$

$$\frac{d^2y}{dx^2} = ?$$

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(-\tan t) = \frac{d}{dt}(-\tan t)\frac{dt}{dx} \\ &= -\frac{1}{\cos^2 t}\left(3a\cos^2 t\sin t\right)^{-1} \\ &= \frac{1}{3a\cos^4 t\sin t}\end{aligned}$$