PC1134 Lecture 3

Topic

Partial Differentiation

Objectives

Understand definition of partial derivative; **Understand** the geometric and physical meanings of partial derivative; **Know** how to calculate partial derivatives of simple functions.

Relevance

Many physical quantities depend on two or more other quantities. Many physical laws (Schrödinger equation, Maxwell's equation, wave equation, diffusion equation, $\cdots$). Partial differentiation is extremely important in a wide range of physical applications.

Partial Differentiation

Consider

$$z = f(x, y)$$

For

$$x \longrightarrow x + \Delta x$$

$$z \longrightarrow z + \Delta z$$

$$\Longrightarrow z + \Delta z = f(x + \Delta x, y)$$

$$\Longrightarrow \Delta z = f(x + \Delta x, y) - f(x, y)$$

$$\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Let $\Delta x \to 0$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$\dfrac{\partial z}{\partial x}$ is the partial derivative of z with respect to x.

Partial Derivative

Partial derivative with respect to y

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} = \lim_{\Delta x \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Other notations:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f'_x = z'_x = \partial_x = \left(\frac{\partial f}{\partial x}\right)_y$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f'_y = z'_y = \partial_y = \left(\frac{\partial f}{\partial y}\right)_x$$

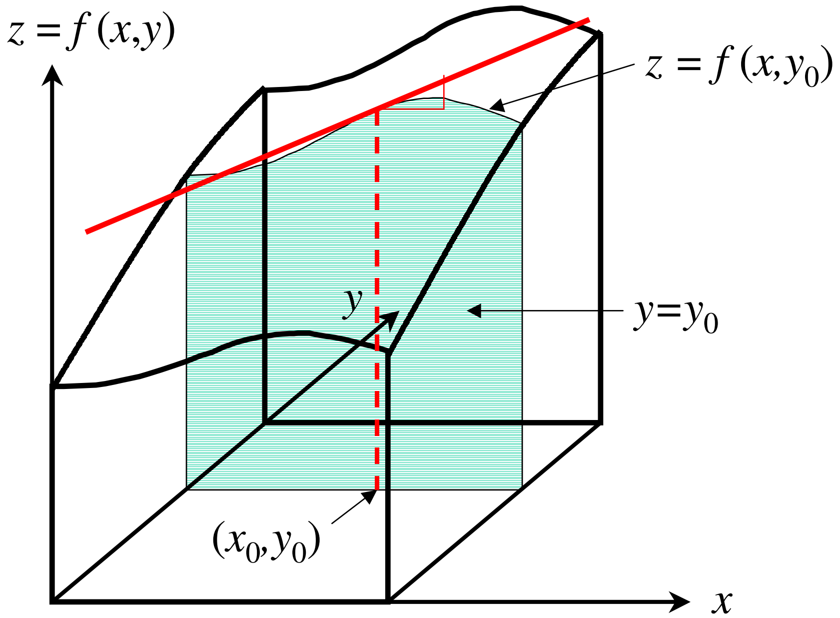
In general, $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are functions of x and y.

$\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at a given point (x_0, y_0) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{x=x_0, y=y_0} = \frac{\partial z}{\partial x}(x_0, y_0)$$

Interpretation of Partial Derivatives

$\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ can be regarded as the "rates of change" of the function along the positive x- and y-directions, respectively.



Intersection of $z = f(x, y)$ and $y = y_0$: $z = f(x, y_0)$.

$f'_x(x_0, y_0)$ is the **slope** or **gradient** of the tangential line of the surface, in the xz-plane and passing through (x_0, y_0). Similarly, $f'_y(x_0, y_0)$ is the **slope** or **gradient** of the tangential line of the surface, in the yz-plane and passing through (x_0, y_0).

Calculate Partial Derivatives

Partial derivatives can be calculated in the same way as ordinary differentiation, since only one of the variables is allowed to change. y is treated as a constant when calculating f'_x for the function $f(x, y)$, and x is treated as a constant when calculating f'_y.

Example 1:

Given $z = x^2 + 3xy + y^2$, calculate the partial derivatives at $(x_0, y_0) = (1, 2)$.

$$\frac{\partial z(x, y)}{\partial x} = 2x + 3y \text{ (y is treated as a constant)}$$

$$\frac{\partial z(x, y)}{\partial y} = 3x + 2y \text{ (x is treated as a constant)}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2 \times 1 + 3 \times 2 = 8$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 3 \times 1 + 2 \times 2 = 7$$

Calculate Partial Derivatives

Example 2:

$$z = x^2 \sin(2y)$$

$$\frac{\partial z}{\partial x} = 2x \sin(2y)$$

$$\frac{\partial z}{\partial y} = x^2 \cos(2y) \cdot 2 = 2x^2 \cos(2y)$$

Example 3:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{r}$$