PC1134 Lecture 4

Topic

Total differentiatial: both x and y are allowed to chagne.

$$\Delta z \Longleftrightarrow (\Delta x, \Delta y)$$

Objectives

Understand the concept of total differentiatial. **Know** the physical and geometric meanings of total differentiatial. **Can** use it in simple applications.

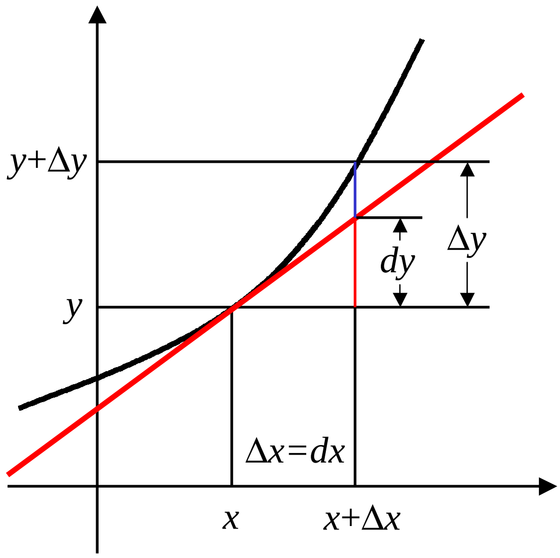
Relevance

Many physical quantities depend on two or more other quantities. Total differential is the change in the function when all variables are allowed to change. It is therefore vital important for calculus of function of two or more variables.

Differential

For $y = f(x)$, define a differential: $dx = \Delta x$

$$dy = \frac{df}{dx} dx \neq \Delta y$$



However, we can write

$$\Delta y = \left(\frac{df}{dx} + \epsilon \right) dx$$

where $\epsilon \to 0$ when $\Delta x \to 0$.

The differential dy is the change in y along the tangent line and is the linear approximation to Δy.

Differential

Consider

$$y = x^2$$

For

$$x \longrightarrow x + \Delta x$$

$$\Delta y = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2$$

$$\frac{dy}{dx} = 2x$$

$$dy = 2xdx$$

$$dy \neq \Delta y$$

$$\Delta y = dy + (\Delta x)^2$$

Total Differential

For $z = f(x, y)$, consider

$$x \longrightarrow x + \Delta x \quad \text{and} \quad y \longrightarrow y + \Delta y$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Add and subtract $f(x + \Delta x, y)$,

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) \\ &\quad + f(x + \Delta x, y) - f(x, y) \end{aligned}$$

Using

$$f(x + \Delta x, y) - f(x, y) = \frac{\partial f}{\partial x}\Delta x + \cdots$$

$$\begin{aligned} & f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) \\ &\qquad = \frac{\partial f(x + \Delta x, y)}{\partial y}\Delta y + \cdots \quad = \quad \frac{\partial f}{\partial y}\Delta y + \cdots \end{aligned}$$

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \cdots$$

Total Differential

From last page,

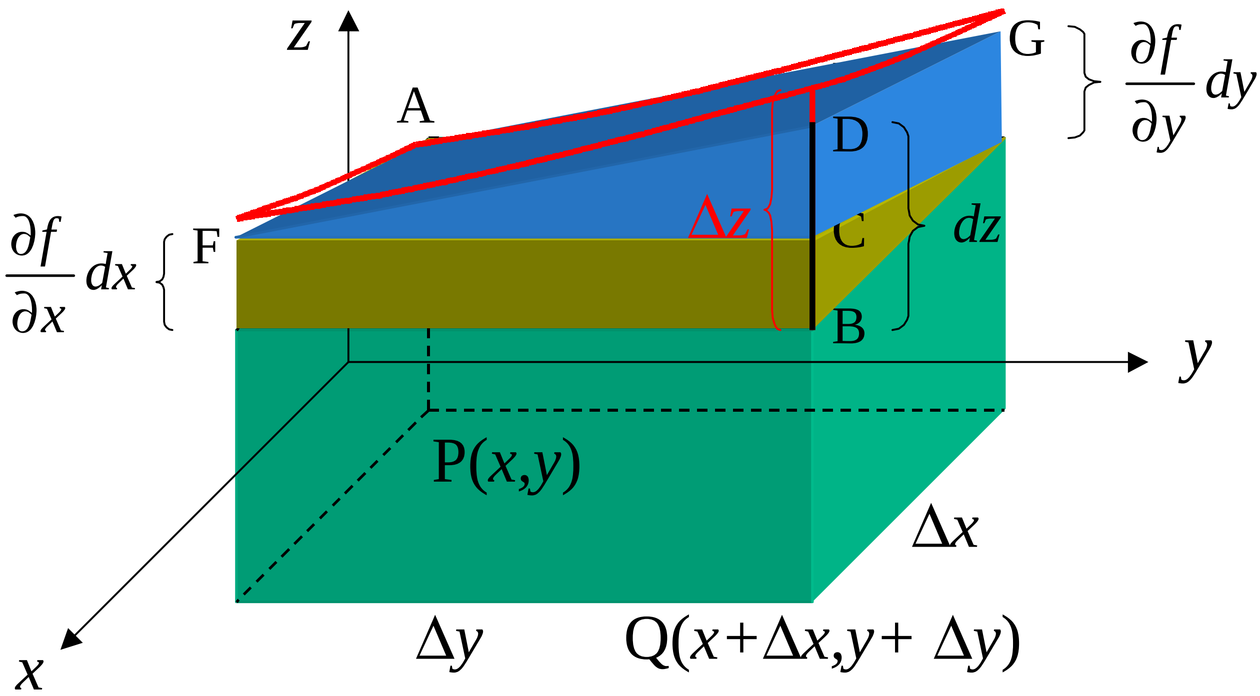
$$\begin{aligned}\Delta z &= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \cdots \\ dz &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\end{aligned}$$

dz is the *total differential*. In general,

$$u = f(x_1, x_2, x_x, \cdots, x_N)$$

$$du = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + \cdots + \frac{\partial f}{\partial x_N}dx_N$$

Geometric Interpretation of Total Differential



Change in z due to change in x: $\dfrac{\partial f}{\partial x} dx$

Change in z due to change in y: $\dfrac{\partial f}{\partial y} dy$

dz is the change in z along the tangent plane when x changes by dx and y by dy.

$$dz \neq \Delta z$$

Example

Estimate $\left[(2.02)^2 + (1.97)^2\right]^{1/3}$

Let

$$\begin{aligned} a &= \left[(2.02)^2 + (1.97)^2\right]^{1/3} \\ &= \left[(2.00 + 0.02)^2 + (2.00 - 0.03)^2\right]^{1/3} \end{aligned}$$

and

$$x = 2.00, \quad \Delta x = 0.02$$
$$y = 2.00, \quad \Delta y = -0.03$$

Consider $z = f(x, y) = \left[(x)^2 + (y)^2\right]^{1/3}$

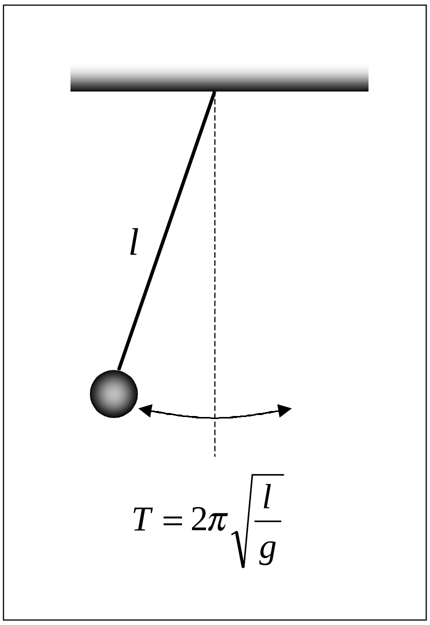
$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$\frac{\partial f}{\partial x} = \frac{2x}{3(x^2 + y^2)^{2/3}}, \qquad \frac{\partial f}{\partial y} = \frac{2y}{3(x^2 + y^2)^{2/3}}$$

$$\begin{aligned} a &\approx \left[x^2 + y^2\right]^{1/3} + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\ &= \left[2^2 + 2^2\right]^{1/3} + \frac{2 \times 2 \times 0.02}{3(2^2 + 2^2)^{2/3}} + \frac{2 \times 2 \times (-0.03)}{3(2^2 + 2^2)^{2/3}} \\ &= 1.99667 \end{aligned}$$

Error Analysis

A simple pendulum is used to measure the acceleration due to gravity, g. The following are obtained: $l = 100.0 \pm 0.1$ cm, $T = 2.000 \pm 0.004$ sec. What is the error in g due to errors in l and T?



$$g = g(l, T) = \frac{4\pi^2 l}{T^2}$$

$$dg = \frac{\partial g}{\partial l} dl + \frac{\partial g}{\partial t} dT = 4\pi^2 \left(\frac{1}{T^2} dl - \frac{2l}{T^3} dT \right)$$

$$dg \leq 4\pi^2 \left(\left| \frac{1}{T^2} \right| |dl| + \left| \frac{2l}{T^3} \right| |dT| \right) = 0.5\pi^2 \text{ cm/s}^2$$