PC1134 Lecture 5

Topic

Rule & Techniques for Partial Differentiation

Objectives

(1) To be able to use the chain rule in calculating partial derivatives; (2) To be able to apply the chain rule to changing of variables, implicit function and other special cases; (3) To understand the concept and be able to calculate total derivative.

Relevance

In this and the next few lectures, we will discuss a few useful rules and techniques of partial differentiation. They are important in calculating derivatives of more complicated multi-variable functions.

Chain Rule

Consider

$$z = f(u, v) \text{ and } \begin{array}{l} u = \phi(x, y) \\ v = \psi(x, y) \end{array}$$

$$\frac{\partial z}{\partial x} = ? \qquad \frac{\partial z}{\partial y} = ?$$

Consider

$$x \to x + \Delta x \Longrightarrow \left. \begin{array}{l} u \to u + \Delta u \\ v \to v + \Delta v \end{array} \right\} z \to z + \Delta z$$

$$\Delta z = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + \cdots$$

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \cdots \qquad \Delta v = \frac{\partial v}{\partial x}\Delta x + \cdots$$

$$\Delta z = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}\Delta x + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\Delta x + \cdots$$

$$\frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \cdots$$

$$\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

Example

Given $z = f(x, y)$ and $\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$

Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}$$

$$\begin{aligned}
& \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 \\
= \quad & \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 + 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \\
& \sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2 - 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \\
= \quad & \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2
\end{aligned}$$

Chain Rule: A Special Case

Given $$z = f(u, x, y) \;\&\; u = \phi(x, y)$$

$$\frac{\partial z}{\partial x} = ? \qquad \frac{\partial z}{\partial y} = ?$$

Assume
$$v = v(x, y) = x$$
$$w = w(x, y) = y$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}$$

Total Derivative

$$z = f(u, v, w)$$

$$u = u(t)$$
$$v = v(t)$$
$$w = w(t)$$

$$z = f(t) = f\left[u(t), v(t), w(t)\right]$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt}$$

$$z = f(u, v, t)$$

$$u = u(t)$$
$$v = v(t)$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial t}$$

Implicit Function

Explicit function:

$$z = f(x, y) \quad \text{or} \quad y = f(x)$$

Implicit function:

$$F(x, y, z) = 0 \quad \text{or} \quad F(x, y) = 0$$

$$F(x, y) = 0 \quad \Longrightarrow \quad y = f(x)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

or

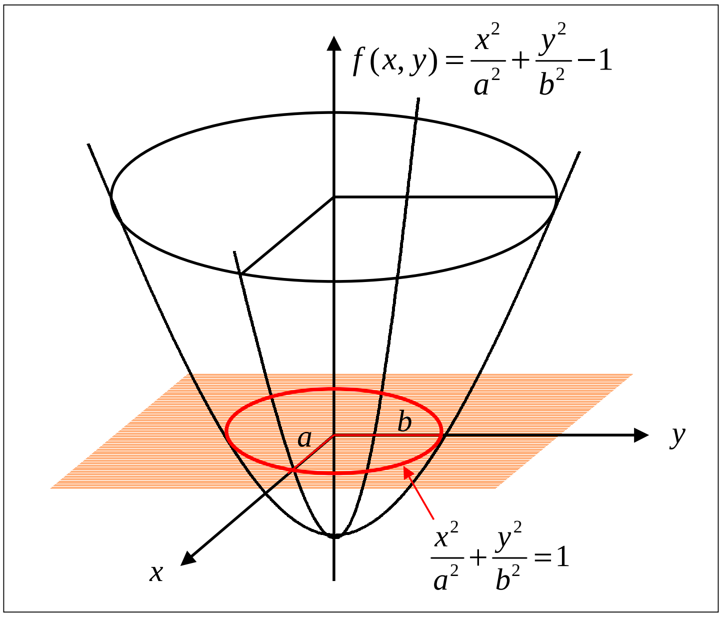
$$F'_x + F'_y \frac{dy}{dx} = 0$$

If $F'_y \neq 0$

$$\frac{dy}{dx} = -\frac{F'_x(x, y)}{F'_y(x, y)}$$

Implicit Function: Example

$$F(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \qquad \frac{dy}{dx} = ?$$



$$\frac{dy}{dx} = -\frac{F'_x(x,y)}{F'_y(x,y)} \qquad F'_x = \frac{2x}{a^2} \qquad F'_y = \frac{2y}{b^2}$$

When $y \neq 0$

$$\frac{dy}{dx} = -\frac{2x/a^2}{2y/b^2} = -\left(\frac{b}{a}\right)^2 \frac{x}{y}$$

Implicit Function

$$F(x,y,z)=0 \quad \Longrightarrow \quad z=f(x,y)$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0$$

If $F_z'(x,y,z)\neq 0$

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}$$

Cyclic Relation

$$F(x,y,z)=0 \quad \begin{array}{ll} \nearrow & x=x(y,z) \\ \rightarrow & y=y(z,x) \\ \searrow & z=z(x,y) \end{array}$$

It can be shown that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{F'_y}{F'_x}$$
$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{F'_z}{F'_y}$$
$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F'_x}{F'_z}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{F'_y}{F'_x}\right)\left(-\frac{F'_z}{F'_y}\right)\left(-\frac{F'_x}{F'_z}\right) = -1$$