

PC1134 Lecture 6

Topic

Higher Derivatives & Taylor Expansion

Objectives

(1) To understand and able to calculate higher derivatives; (2) To understand Taylor series and able to calculate the first few terms for a function of one variable and at least the linear terms for function of two variables.

Relevance

Dependence of one physical quantity on others can be expanded in power series and different level approximation can be obtained by keeping certain terms in the expansion. Example: harmonic oscillation.

Higher Derivatives

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = f'_x(x, y)$$

$$\frac{\partial z}{\partial y} = f'_y(x, y)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x, y)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x} = f''_{yx}(x, y)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x, y)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = f''_{yy}(x, y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \quad (\text{if continuous})$$

Example

$$z = x^3y^2 - 3xy^3 - xy + 1$$

$$\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$$

$$\frac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x$$

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 - 1$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x^2y - 9y^2 - 1$$

$$\frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy$$

Expanding Functions in Power Series

Using $\sin x$ as an example,

$$\sin x = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

$$\sin 0 = 0 \implies a_0 = 0$$

Differentiate both sides

$$\cos x = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

$$\cos 0 = 1 \implies a_1 = 1$$

$$-\sin x = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \cdots$$

$$\sin 0 = 0 \implies a_2 = 0$$

.....

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Maclaurin Series or Taylor Series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots$$

$$f'(x) = a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1} + \cdots$$

$$f''(x) = 2a_2 + \cdots + n(n-1)a_n(x - x_0)^{n-2} + \cdots$$

.....

$$f^{(n)}(x) = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 a_n + \cdots$$

Let $x = x_0$

$$\Rightarrow \begin{array}{lll} f(x_0) = a_0 & f'(x_0) = a_1 & f''(x_0) = 2a_2 \\ f'''(x_0) = 3!a_3 & \cdots & f^{(n)}(x_0) = n!a_n \end{array}$$

Taylor series about $x = x_0$:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \cdots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) + \cdots$$

Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

Power Series for $f(x, y)$

$$\begin{aligned} f(x, y) = & a_{00} + a_{10}(x - x_0) + a_{01}(y - y_0) + a_{20}(x - x_0)^2 + \\ & a_{11}(x - x_0)(y - y_0) + a_{02}(y - y_0)^2 + \\ & a_{30}(x - x_0)^3 + a_{21}(x - x_0)^2(y - y_0) + \\ & a_{12}(x - x_0)(y - y_0)^2 + a_{03}(y - y_0)^3 + \dots \end{aligned}$$

$$f'_x = a_{10} + 2a_{20}(x - x_0) + a_{11}(y - y_0) + \dots$$

$$f'_y = a_{01} + a_{11}(x - x_0) + 2a_{02}(y - y_0) + \dots$$

$$f''_{xx} = 2a_{20} + \dots$$

$$f''_{xy} = a_{11} + \dots$$

$$f''_{yy} = 2a_{02} + \dots$$

Let $x = x_0$ and $y = y_0$,

$$f(x_0, y_0) = a_{00}$$

$$f'_x(x_0, y_0) = a_{10}$$

$$f'_y(x_0, y_0) = a_{01}$$

$$f''_{xx}(x_0, y_0) = 2a_{20}$$

$$f''_{xy}(x_0, y_0) = a_{11}$$

$$f''_{yy}(x_0, y_0) = 2a_{02}$$

Power Series for $f(x, y)$ (cont.)

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + f'_x(x - x_0) + f'_y(y - y_0) + \\ & \frac{1}{2!} \left[f''_{xx}(x - x_0)^2 + 2f''_{xy}(x - x_0)(y - y_0) + \right. \\ & \left. f''_{yy}(y - y_0)^2 \right] + \dots \end{aligned}$$

If $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$, then

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + f'_x \Delta x + f'_y \Delta y + \\ & \frac{1}{2!} \left[f''_{xx}(\Delta x)^2 + 2f''_{xy}(\Delta x)(\Delta y) + f''_{yy}(\Delta y)^2 \right] + \dots \end{aligned}$$

Using the notation

$$\begin{aligned} \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y &= \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right) f(x, y) \\ \frac{1}{2!} \left[\frac{\partial^2 f}{\partial x^2} (\Delta x)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f}{\partial y^2} (\Delta y)^2 \right] \\ &= \frac{1}{2!} \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right)^2 f(x, y) \\ f(x, y) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right)^n f(x, y) \right]_{x_0, y_0} \end{aligned}$$

Function of Many Variables

$$f(x_1, x_2, \dots, x_n)$$

Let

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

$$\begin{aligned} f(\mathbf{x}) = & f(\mathbf{x}_0) + \sum_i \frac{\partial f}{\partial x_i} \Delta x_i + \\ & \frac{1}{2!} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + \dots \end{aligned}$$

Or

$$f(\mathbf{x}) = \sum_0^{\infty} \frac{1}{n!} [(\Delta \mathbf{x} \cdot \nabla)^n f(\mathbf{x})]_{\mathbf{x}=\mathbf{x}_0}$$