PC1134 Lecture 8

Topic

Propagation of Error

Objectives

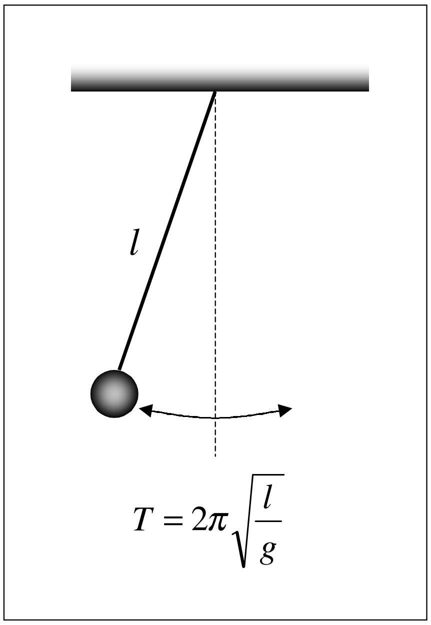
Understand how errors in directly measured quantities effect the uncertainty of a quantity calculated from these primary quantities; **know** how to estimate the error in the calculated quantity.

Relevance

In many cases, we don't measure the physical quantity directly, instead we measure several primary quantities first, then we calculate the quantity we wanted from these primary quantities. The error in this physical quantity is caused by the uncertainty in those primary quantities and we need to know how to estimate the uncertainty of this physical quantity.

Introduction

The acceleration due to gravity can by measured using a simple pendulum (see figure below) by measuring the length of the string (l) and the period of small oscillation (T).



g can be calculated from

$$g = g(l, T) = \frac{4\pi^2 l}{T^2}$$

What is the uncertainty in g and how it is related to uncertainties in l and T?

Propagation of Errors

Assume that the quantities $x_1, x_2, \cdots, x_n$ can be measured directly and y is calculated from

$$y = f(x_1, x_2, \cdots, x_n)$$

If Δx_i is the error in a measurement of x_i, then the error in y is given by

$$\Delta y = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$

Further assume that each quantity is measured m times, and $x_{i1}, x_{i2}, \cdots, x_{im}$ are obtained for x_i.

Based on the average of each measured quantity

$$\bar{x}_i = \frac{1}{m}\sum_{k=1}^{m} x_{ik}$$

we can calculate a y

$$y_0 = f(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$$

Propagation of Errors

Let the deviations of the measured values for x_i from their average $\bar{x}_i$ be $\Delta x_{i1}, \Delta x_{i2}, \cdots, \Delta x_{im}$, respectively. We have m deviations in y from y_0.

$$
\begin{aligned}
\Delta y_1 &= \frac{\partial f}{\partial x_1}\Delta x_{11} + \frac{\partial f}{\partial x_2}\Delta x_{21} + \cdots + \frac{\partial f}{\partial x_n}\Delta x_{n1} \\
\Delta y_2 &= \frac{\partial f}{\partial x_1}\Delta x_{12} + \frac{\partial f}{\partial x_2}\Delta x_{22} + \cdots + \frac{\partial f}{\partial x_n}\Delta x_{n2} \\
&\cdots\cdots \\
\Delta y_m &= \frac{\partial f}{\partial x_1}\Delta x_{1m} + \frac{\partial f}{\partial x_2}\Delta x_{2m} + \cdots + \frac{\partial f}{\partial x_n}\Delta x_{nm}
\end{aligned}
$$

Add these m equations, we have

$$
\sum_{k=1}^{m} \Delta y_k = \frac{\partial f}{\partial x_1}\sum_{k=1}^{m}\Delta x_{1k} + \frac{\partial f}{\partial x_2}\sum_{k=1}^{m}\Delta x_{2k} + \cdots
$$

$$
+\frac{\partial f}{\partial x_n}\sum_{k=1}^{m}\Delta x_{nk}
$$

Propagation of Errors

If m is large enough,

$$\sum_{k=1}^{m} \Delta x_{ik} = 0$$

Thus

$$\sum_{k=1}^{m} \Delta y_k = 0$$

i.e.

$$\sum_{k=1}^{m} (y_k - y_0) = \sum_{k=1}^{m} y_k - m y_0 = 0$$

Therefore

$$y_0 = \frac{1}{m} \sum_{k=1}^{m} y_k$$

or

$$\bar{f}(x_1, x_2, \cdots, x_n) = f(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$$

The average value of an indirectly measured quantity (y) can be taken as the value calculated using the average values of the directly measured quantities.

Propagation of Errors

Square both sides of each equation on slide 4 and then add the m equations, we have

$$\sum_{k=1}^{m}(\Delta y_k)^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sum_{k=1}^{m}(\Delta x_{1k})^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sum_{k=1}^{m}(\Delta x_{2k})^2$$

$$+ \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sum_{k=1}^{m}(\Delta x_{nk})^2$$

$$+2\sum_{i<j}\left[\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right)\sum_{k=1}^{m}\Delta x_{ik}\Delta x_{jk}\right]$$

Divide both sides by $(m-1)$ and make use of the definition of standard deviation, we obtain the law of error propagation

$$\sigma_y^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2$$

$$+2\sum_{i<j}\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right)K_{ij}$$

where

$$K_{ij} = \frac{1}{m-1}\sum_{k=1}^{m}\Delta x_{ik}\Delta x_{jk}$$

Propagation of Errors

K_{ij} indicates the correlation between uncertainties in x_i and x_j. If errors in x_i are independent of each other, then $K_{ij} = 0$ and

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2}$$

This is called the law of propagation of standard deviation.

Standard Deviation of the Mean

The standard deviation we discussed so far describes the uncertainty of a single measurement, even though it is based on the analysis of N measurements.

If we have N measured values, we can calculated the mean, $\bar{x}$ which is more accurate than any of the individual measurements.

However, $\bar{x}$ is still not the real value. What is the uncertainty in $\bar{x}$ then?

Since

$$\bar{x} = \frac{1}{N}(x_1 + x_2 + \cdots + x_N)$$

$$\begin{aligned}
\sigma_{\bar{x}}^2 &= \left(\frac{\partial \bar{x}}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial \bar{x}}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial \bar{x}}{\partial x_N}\right)^2 \sigma_{x_N}^2 \\
&= \frac{1}{N^2}\left(\sigma_{x_1}^2 + \sigma_{x_2}^2 + \cdots + \sigma_{x_N}^2\right)
\end{aligned}$$

Because x_1, $x_2, \cdots, x_N$ are measurement of the same quantity, $\sigma_{x_1} = \sigma_{x_2} = \cdots = \sigma_{x_N} = \sigma$.

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{N}$$

Standard Deviation of the Mean

Thus

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

Random error can be reduced by performing multiple measurements and averaging the results.

However, the error reduction is proportional to $1/\sqrt{N}$. When $N > 10 \sim 20$, the reduction becomes very slow.

Data Processing

For measurement of a single quantity

1. Repeat measurement N times. Assume the measured values are $x_1, x_2, \cdots, x_N$.

2. Calculate the mean

$$\bar{x} = \frac{1}{N}(x_1 + x_2 + \cdots + x_N)$$

3. Calculate the standard deviation

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

4. Calculate the standard deviation of the mean

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

5. The result of the measurement is written as

$$\bar{x} \pm \sigma_{\bar{x}}$$

Data Processing

For a quantity measured indirectly

1. Repeat measurement of each primary quantity m times. Calculate the mean and standard deviation of each primary quantity,

$$\bar{x}_i = \frac{1}{m}(x_{i1} + x_{i2} + \cdots + x_{im})$$

$$\sigma_{x_i} = \sqrt{\frac{1}{m-1}\sum_{k=1}^{N}(x_{ik} - \bar{x}_i)^2}$$

2. The best estimate for the quantity to be measured is given by $\bar{y} = f(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N)$.

3. Calculate the standard deviation in y

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2}$$

4. The result is given by

$$\bar{y} \pm \sigma_y$$