PC1134 Lecture 9

Topic

Stationary Values of Functions

Objectives

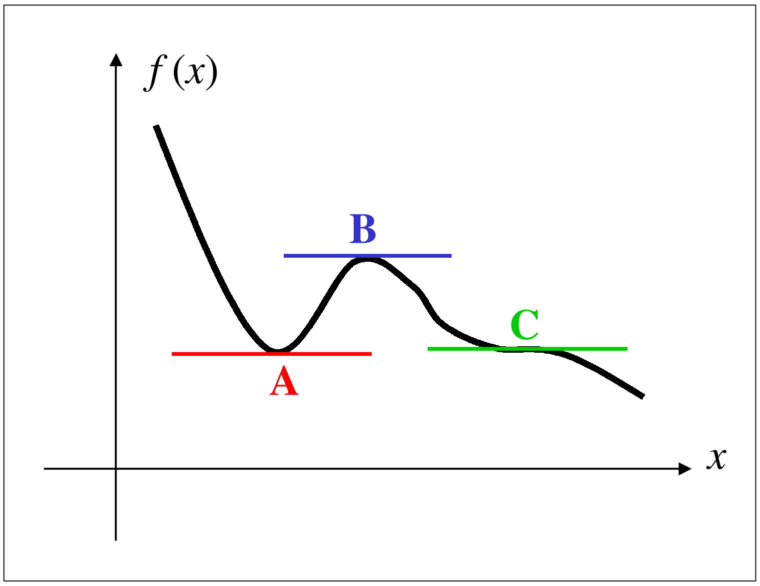
(1) To understand the nature of stationary points of a function; (2) To be able to find the stationary points of given functions; (3) To be able to apply it to the maximization or minimization problems.

Relevance

Optimization (maximization or minimization) is widely used not only in scientific problems but also in other areas. Even though the quantity to be optimized and the system are different, the method is the same.

Stationary Point of $f(x)$

$$y = f(x)$$



$$\frac{dy}{dx} = 0$$

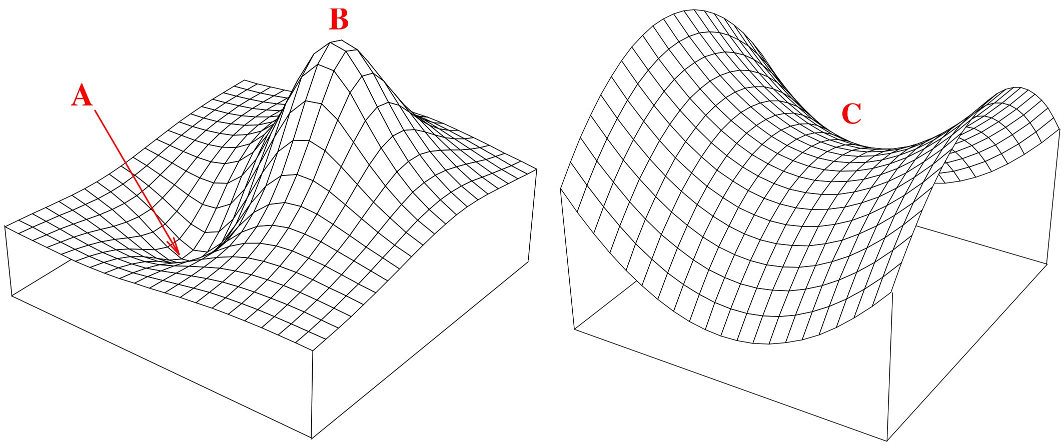
$$\Downarrow$$

Stationary Point

$$\overbrace{\begin{array}{ccc} \text{maximum} & \text{minimum} & \text{point of inflection} \\ \dfrac{d^2 f}{dx^2} < 0 & \dfrac{d^2 f}{dx^2} > 0 & \dfrac{d^2 f}{dx^2} = 0 \end{array}}$$

Stationary Point of $f(x,y)$

$$z = f(x,y)$$



If $z_{\max} = f(x_0, y_0)$, then

$z = f(x, y_0)$ has a maximum at x_0

$z = f(x_0, y)$ has a maximum at y_0

$$\left.\frac{\partial z}{\partial x}\right|_{x_0} = 0, \qquad \left.\frac{\partial z}{\partial y}\right|_{y_0} = 0.$$

$$\frac{\partial z}{\partial x} = 0 \;\&\; \frac{\partial z}{\partial y} = 0$$

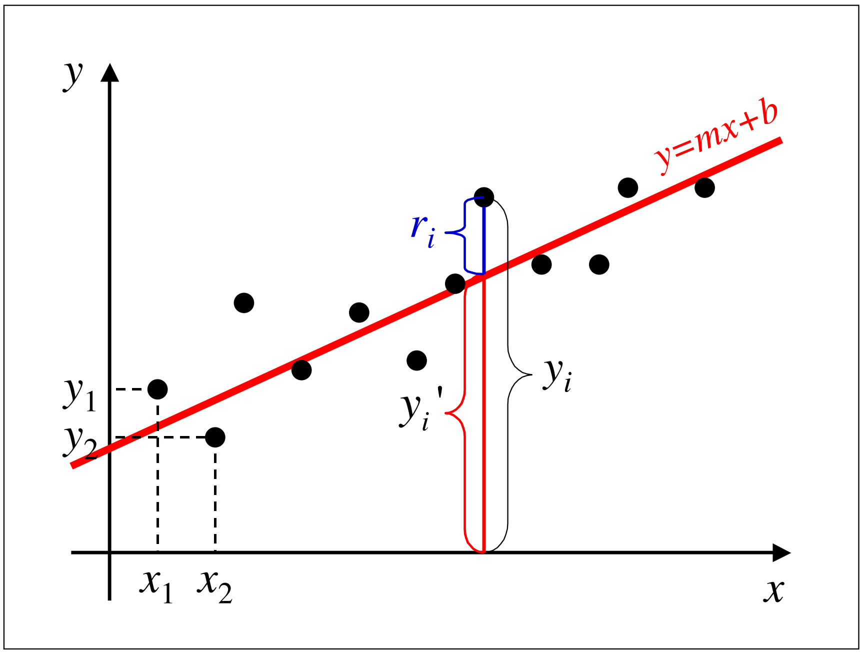
$$\Downarrow$$

Stationary Point

$$\overbrace{\text{maximum} \quad \text{minimum} \quad \text{point of inflection}}$$

Curve Fitting

Measured Data		Fit to $y = mx + b$
x_1	y_1	$y_1' = mx_1 + b$
x_2	y_2	$y_2' = mx_2 + b$
$\cdots$	$\cdots$	$\cdots$
x_i	y_i	$y_i' = mx_i + b$
$\cdots$	$\cdots$	$\cdots$
x_N	y_N	$y_N' = mx_N + b$



$$r_i = y_i' - y_i = mx_i + b - y_i$$

$$r_1 = y_1' - y_1 = mx_1 + b - y_1 \quad (i = 1)$$

Curve Fitting (cont.)

Good Fit $\Longleftrightarrow$ Small r_i

$$R \equiv r_1^2 + r_2^2 + \cdots + r_N^2 = \sum_{i=1}^{N} r_i^2 = \sum_{i=1}^{N} (mx_i + b - y_i)^2$$

Good Fit $\Longleftrightarrow$ Small R

Goal: minimize R!

$$R = f(m, b)$$

$$\frac{\partial R}{\partial m} = 0 = 2\sum_{i=1}^{N} (mx_i + b - y_i)\, x_i$$

$$\frac{\partial R}{\partial b} = 0 = 2\sum_{i=1}^{N} (mx_i + b - y_i)$$

$$\left\{ \begin{aligned} \left(\sum_{i=1}^{N} x_i^2\right) m + \left(\sum_{i=1}^{N} x_i\right) b &= \sum_{i=1}^{N} x_i y_i \\ \left(\sum_{i=1}^{N} x_i\right) m + Nb &= \sum_{i=1}^{N} y_i \end{aligned} \right.$$

Curve Fitting (cont.)

Solve the equations for m and b, we have

$$m = \frac{N\left(\sum_{i=1}^{N} x_i y_i\right) - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)}{N\left(\sum_{i=1}^{N} x_i^2\right) - \left(\sum_{i=1}^{N} x_i\right)^2}$$

$$b = \frac{\left(\sum_{i=1}^{N} x_i^2\right)\left(\sum_{i=1}^{N} y_i\right) - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} x_i y_i\right)}{N\left(\sum_{i=1}^{N} x_i^2\right) - \left(\sum_{i=1}^{N} x_i\right)^2}$$