# PC1134 Lecture 11

## Topic

Stationary values under constraints

## Objectives

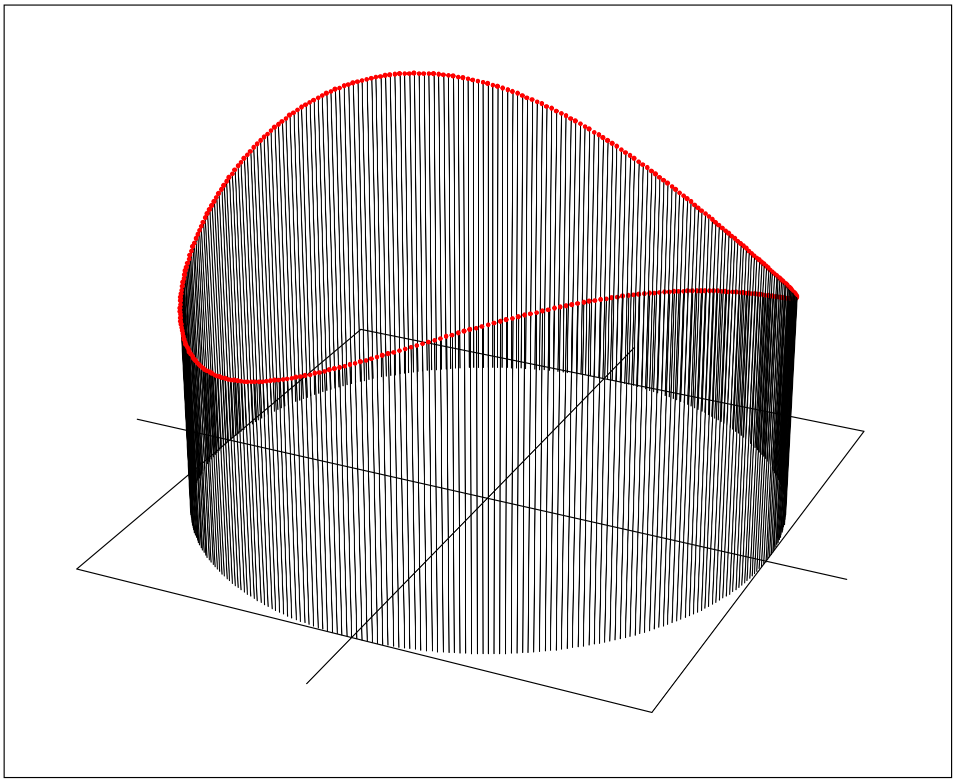
To be able to use the Lagrange multiplier method to find the stationary points of a given function with constraints on the variables.

## Relevance

This is an extension of the last lecture. The variables are normally subject to certain constraints when the maxima and/or minima of a function are being sort.

# Introduction

The temperature of a point $(x, y)$ on a unit circle is given by $T(x, y) = xy$. Find the temperature of the two hottest points on this circle.



To find the maximum of

$$T(x, y) = xy$$

$x$ and $y$ are subject to the constraint:

$$x^2 + y^2 = 1$$

# Lagrange Multipliers

To find the maximum/minimum of $z = f(x, y)$

$x$ and $y$ are subject to the constraint: $g(x, y) = c$

For $f(x, y)$ to have a stationary point

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0$$

$$\Longrightarrow \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{1}$$

Differentiate both sides of Eq.(2)

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \tag{2}$$

Multiplying $dg$ by $\lambda$ and adding to $df$

$$df + \lambda dg = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right) dy = 0$$

$$\left.\begin{array}{r} \dfrac{\partial f}{\partial x} + \lambda\dfrac{\partial g}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} + \lambda\dfrac{\partial g}{\partial y} = 0 \\ g(x, y) = c \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} x = ? \\ y = ? \\ \lambda = ? \end{array}\right.$$

# Systematic Approach

Consider

$$F(x,y) = f(x,y) + \lambda g(x,y) \tag{3}$$

The stationary point of $F(x,y)$ is given by

$$\begin{aligned}
\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} &= 0 \\
\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} &= 0 \\
g(x,y) &= c
\end{aligned}$$

These are the same equations as in the last slide. But $x$ and $y$ are indenpendent variables now.

To find the maximum or minimum of $f(x,y)$ where $x$ and $y$ are related by $g(x,y) = c$, (1) form the function $F(x,y)$; (2) calculate and set the two partial derivatives of $F(x,y)$ to zero; (3) solve these equations and the equation $g(x,y) = c$ for the three unknown $x$, $y$ and $\lambda$.

# Example

The temperature of a point $(x, y)$ on a unit circle is given by $T(x, y) = xy$. Find the temperature of the two hottest points on this circle.

To maximize $T(x, y)$, subject to the constraint $g(x, y) = x^2 + y^2 = 1$.

Consider

$$F(x, y) = T(x, y) + \lambda g(x, y) = xy + \lambda(x^2 + y^2)$$

Let

$$\frac{\partial F}{\partial x} = y + 2\lambda x = 0$$

$$\frac{\partial F}{\partial y} = x + 2\lambda y = 0$$

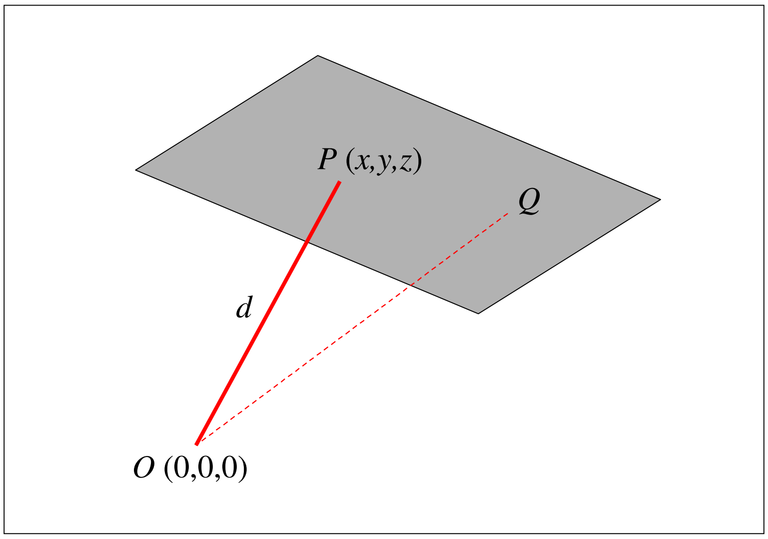
Solving these two equations together with $x^2 + y^2 = 1$, we get four possible stationary pionts of $T(x, y)$ at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$

$$\begin{aligned} T(1/\sqrt{2}, 1/\sqrt{2}) = T(-1/\sqrt{2}, -1/\sqrt{2}) &= 1/2 \text{ (max)} \\ T(1/\sqrt{2}, -1/\sqrt{2}) = T(-1/\sqrt{2}, 1/\sqrt{2}) &= -1/2 \end{aligned}$$

# Example

Find the shortest distance from the origin to the plane

$$x - 2y - 2z = 3$$




To minimize distance: $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
$x$, $y$ and $z$ subject to the constraint $x - 2y - 2z = 3$

Or equivalently
To minimize

$$f(x, y, z) = d^2 = x^2 + y^2 + z^2$$

Subject to the constraint

$$\phi(x, y, z) = x - 2y - 2z = 3$$

# Example (cont.)

$$f(x, y, z) = d^2 = x^2 + y^2 + z^2$$

$$\phi(x, y, z) = x - 2y - 2z = 3$$

Let

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(x - 2y - 2z)$$

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2x + \lambda = 0 \\
\frac{\partial F}{\partial y} &= 2y - 2\lambda = 0 \\
\frac{\partial F}{\partial z} &= 2z - 2\lambda = 0 \\
x - 2y - 2z &= 3
\end{aligned}$$

The stationary point is given by the solution of these four equations

$$x = \frac{1}{3}, \quad y = z = -\frac{2}{3}$$

$$d_{\min} = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = 1$$

# General Case

to max/min : $f(x_1, x_2, \cdots, x_n)$

constraints : $\begin{cases} \phi_1(x_1, x_2, \cdots, x_n) = 0 \\ \phi_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots \\ \phi_m(x_1, x_2, \cdots, x_n) = 0 \end{cases}$

$$F(x_1, x_2, \cdots, x_n) = f + \lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_m\phi_m$$

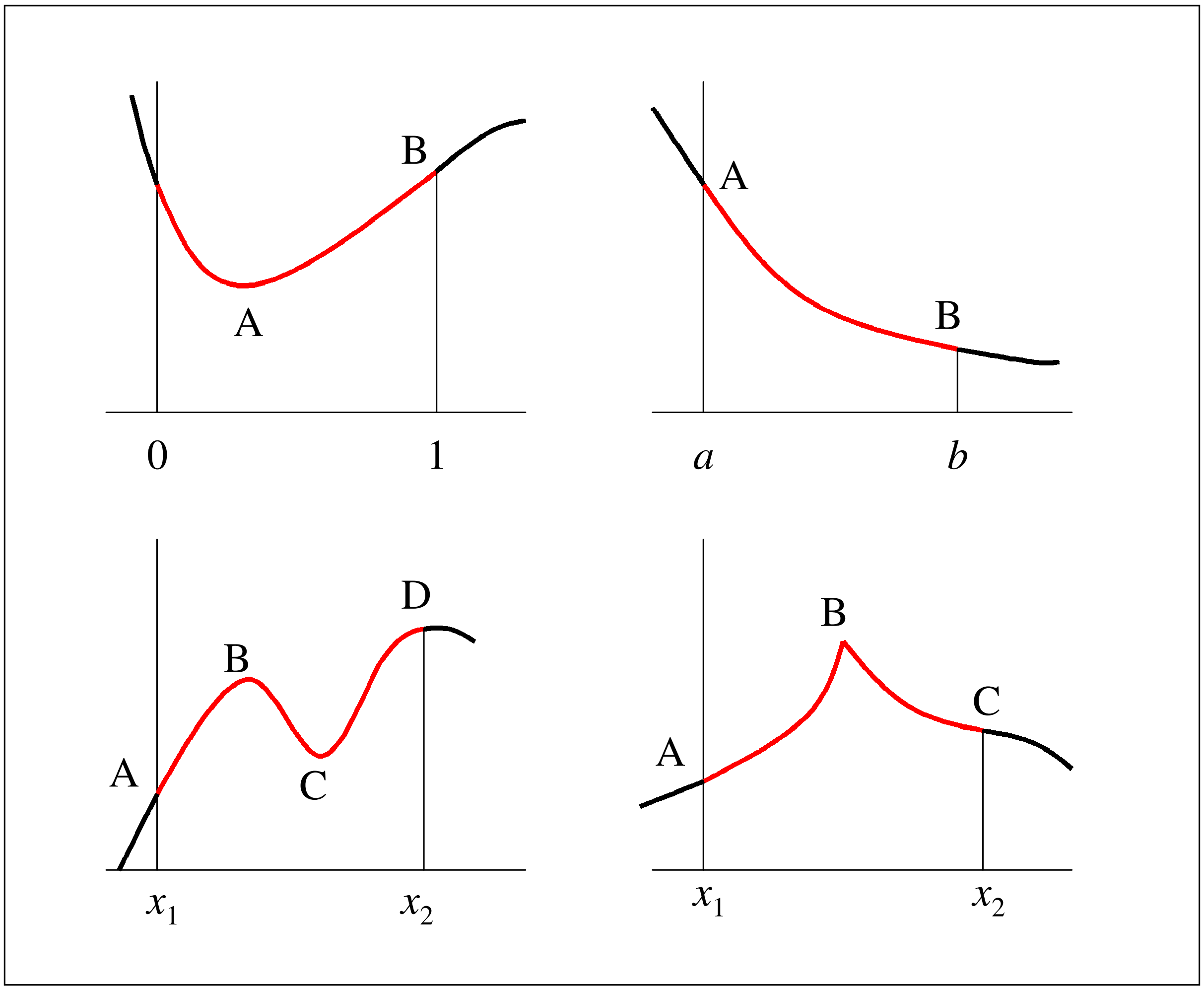
$$\begin{cases} \dfrac{\partial F}{\partial x_1} = \dfrac{\partial f}{\partial x_1} + \lambda_1\dfrac{\partial \phi_1}{\partial x_1} + \lambda_2\dfrac{\partial \phi_2}{\partial x_1} + \cdots + \lambda_m\dfrac{\partial \phi_m}{\partial x_1} = 0 \\ \dfrac{\partial F}{\partial x_2} = \dfrac{\partial f}{\partial x_2} + \lambda_1\dfrac{\partial \phi_1}{\partial x_2} + \lambda_2\dfrac{\partial \phi_2}{\partial x_2} + \cdots + \lambda_m\dfrac{\partial \phi_m}{\partial x_2} = 0 \\ \cdots\cdots \\ \dfrac{\partial F}{\partial x_n} = \dfrac{\partial f}{\partial x_n} + \lambda_1\dfrac{\partial \phi_1}{\partial x_n} + \lambda_2\dfrac{\partial \phi_2}{\partial x_n} + \cdots + \lambda_m\dfrac{\partial \phi_m}{\partial x_n} = 0 \end{cases}$$

These $n$ equations and the $m$ constraints together can be solved for the $n+m$ variables:

$$x_1, x_2, \cdots, x_n, \lambda_1, \lambda_2, \cdots, \lambda_m$$

# Endpoints & Boundaries

Calculation fails if max/min is at an endpoint or at a boundary.




Must consider end points and boundaries as well as points predicted by calculus.

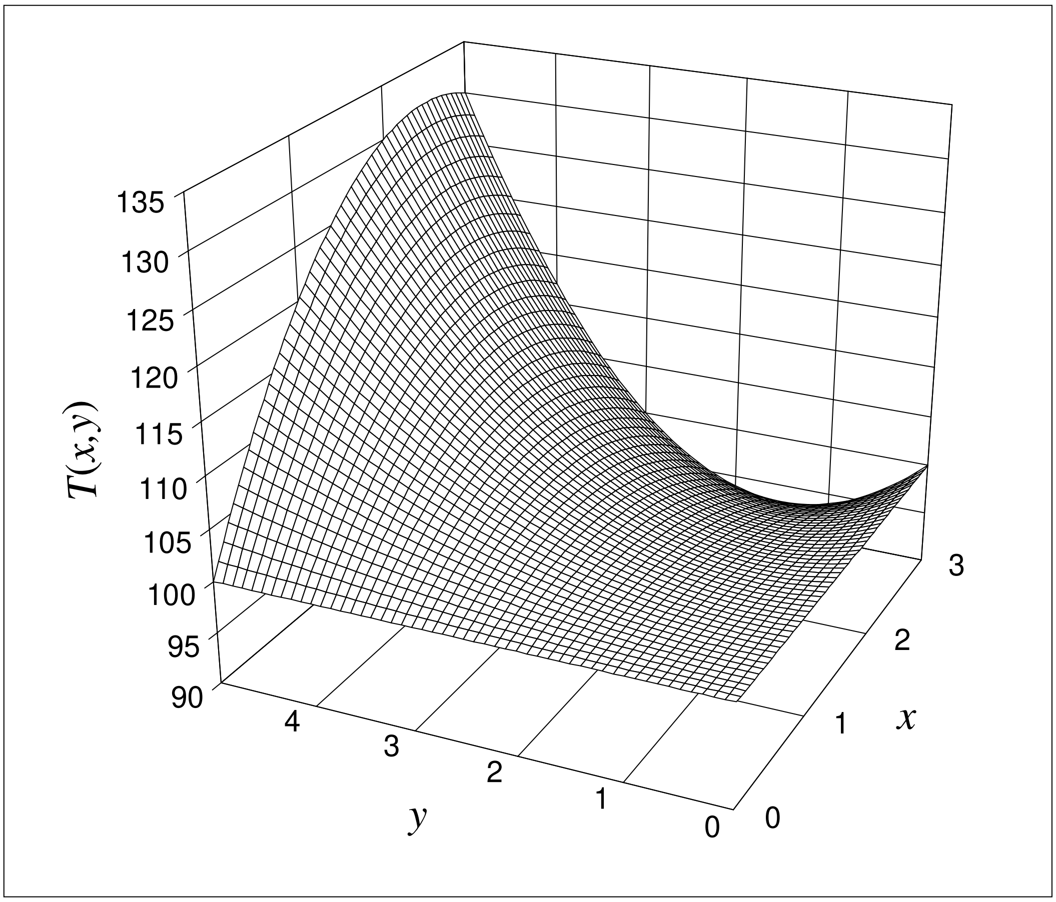
Study function case by case (plot by computer).

# Example

The temperature in a rectangular plate bounded by $x = 0$, $y = 0$, $x = 3$ and $y = 5$ is

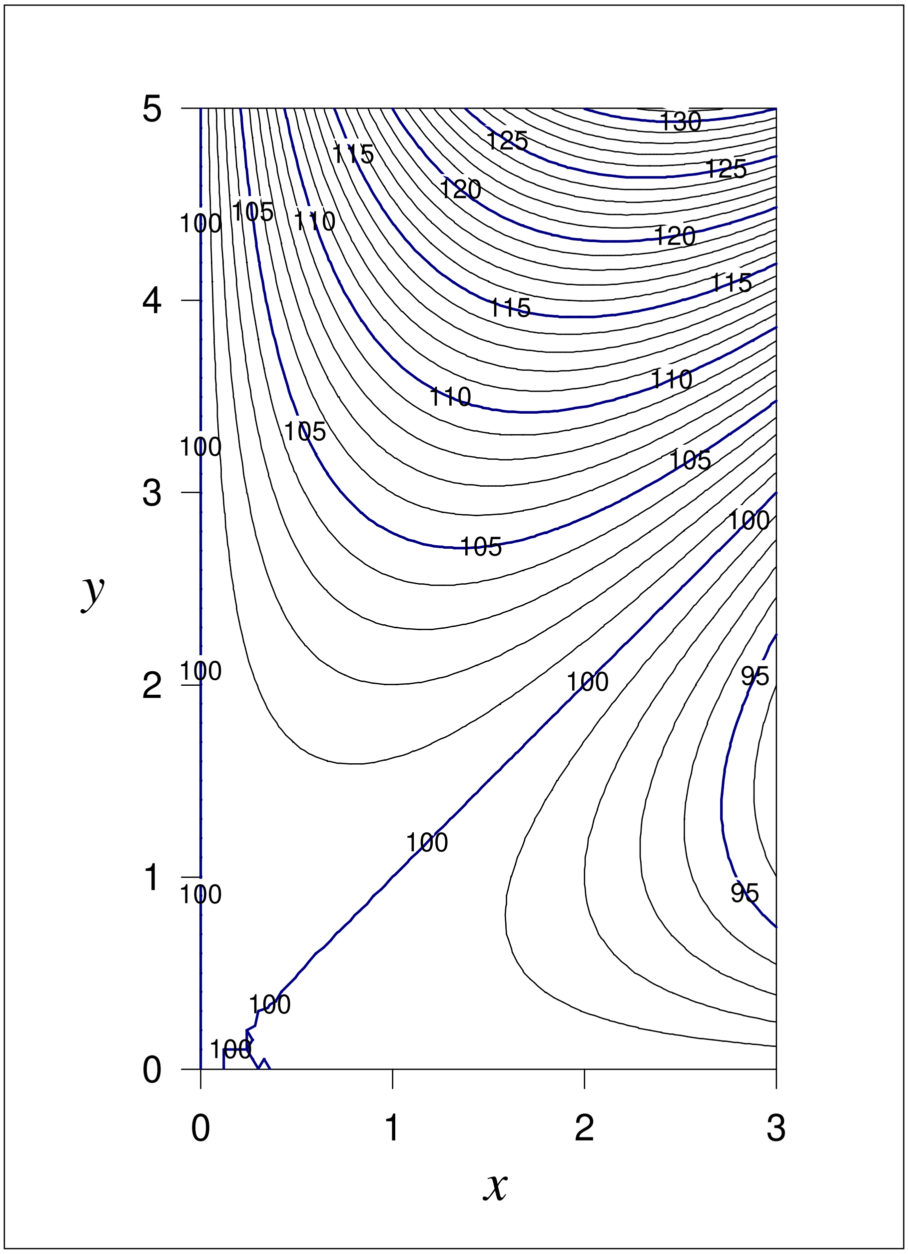
$$T = xy^2 - x^2y + 100$$

Find the hottest and coldest points of the plate.



# Example (cont.)

$$T = xy^2 - x^2y + 100$$



# Example (cont.)

$T = xy^2 - x^2y + 100$

$$\begin{cases} \partial T/\partial x = y^2 - 2xy = 0 \\ \partial T/\partial y = 2xy - x^2 = 0 \end{cases} \quad \Longrightarrow \quad \begin{array}{c} (x, y) = (0, 0) \\ T(0, 0) = 100 \end{array}$$

Along the boundary $x = 0$: $T(0, y) = 100$.
Along the boundary $y = 0$: $T(x, 0) = 100$.
Along the boundary $x = 3$:

$$T(3, y) = 3y^2 - 9y + 100$$

$$\frac{dT}{dy} = 6y - 9 = 0 \quad \Longrightarrow \quad y = \frac{3}{2} \;\&\; T\left(3, \frac{3}{2}\right) = 93.25$$

Along the boundary $y = 5$:

$$T(x, 5) = 25x - 5x^2 + 100$$

$$\frac{dT}{dx} = 25-10x = 0 \quad \Longrightarrow \quad x = \frac{5}{2} \;\&\; T\left(\frac{5}{2}, 5\right) = 131.25$$

Corners:

$$T(0, 0) = T(0, 5) = T(3, 0) = 100, T(3, 5) = 130$$

Hottest point: (5/2,5), $T = 131.25$
Coldest piont: (3,3/2), $T = 93.25$