PC1134 Lecture 13

Topic

Triple products of vectors

Objectives

To understand and be able to calcualte triple scalar and triple vector products of three vectors.

Example of Application

- Torque about a point & torque about a line
- Angular momentum
- Centripetal acceleration of a point in a rotating object

Triple Scalar Product

$$\vec{A}\cdot(\vec{B}\times\vec{C})$$

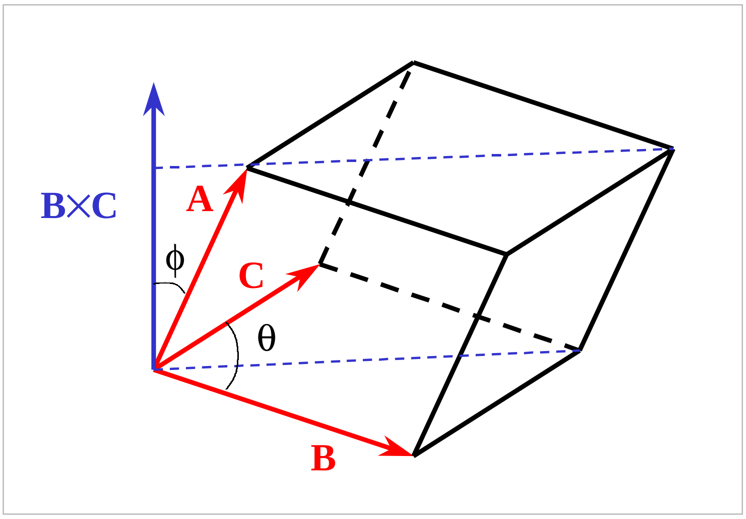
$$\begin{aligned}
\vec{B}\times\vec{C} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\
&= (B_yC_z - B_zC_y)\hat{x} + (B_zC_x - B_xC_z)\hat{y} + \\
&\quad (B_xC_y - B_yC_x)\hat{z}
\end{aligned}$$

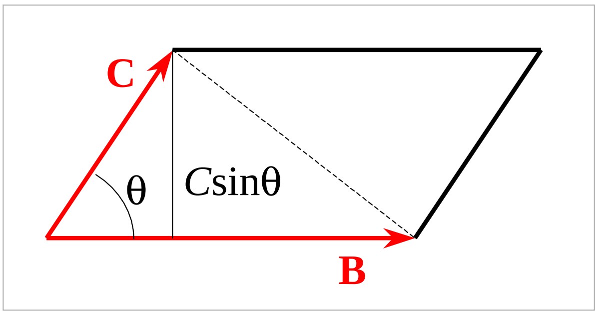
$$\begin{aligned}
&\vec{A}\cdot(\vec{B}\times\vec{C}) \\
&= A_x(B_yC_z - B_zC_y) + \\
&\quad A_y(B_zC_x - B_xC_z) + \\
&\quad A_z(B_xC_y - B_yC_x) \\
&= A_xB_yC_z + A_yB_zC_x + A_zB_xC_y \\
&\quad -A_xB_zC_y - A_yB_xC_z - A_zB_yC_x
\end{aligned}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Triple Scalar Product

$$V = \vec{A} \cdot (\vec{B} \times \vec{C})$$

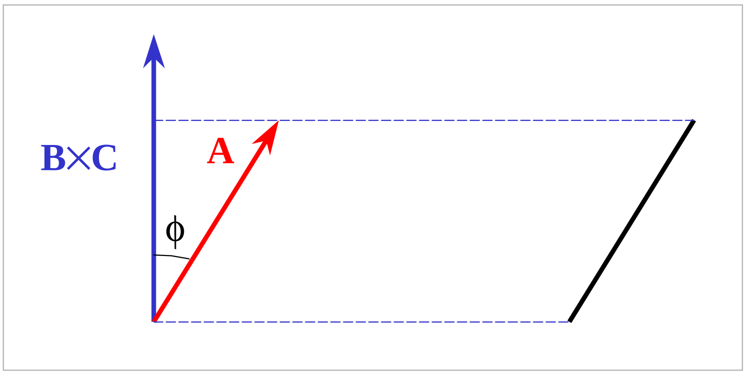




Base area: $|\vec{B} \times \vec{C}|$

Height of the parallelepiped:

$$A \cos \phi$$

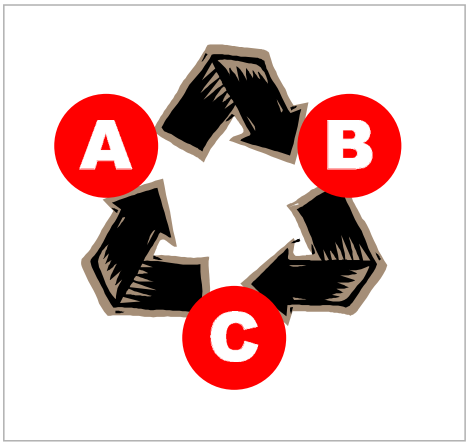


$$\text{Volume} = |\vec{B} \times \vec{C}| A \cos \phi = \vec{A} \cdot (\vec{B} \times \vec{C})$$

Triple Scalar Product

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C})$$

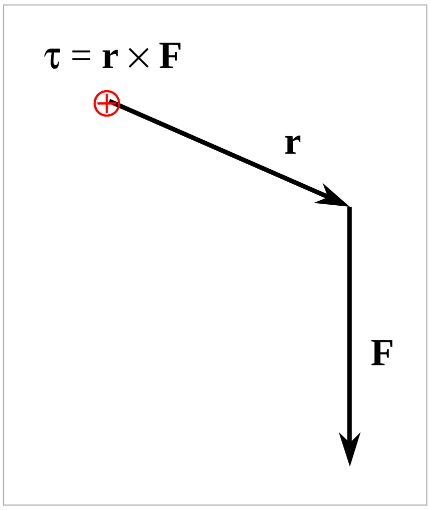
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

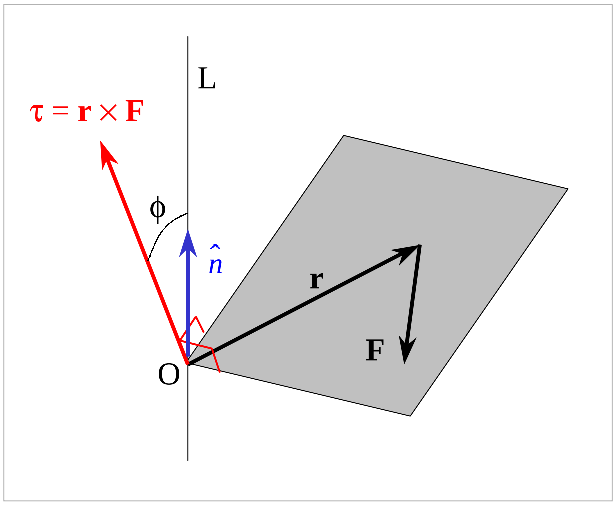


$$\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{A} \cdot (\vec{C} \times \vec{B}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$$

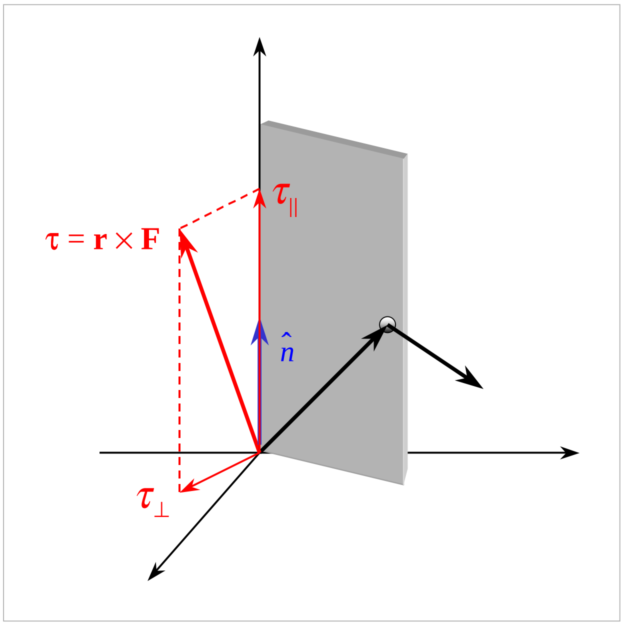
Application of Triple Scalar Product

$\vec{\tau}$ is perpendicular to the paper and points inward.





In general, $\vec{r} \times \vec{F}$ is the torque about point O. It may not be parallel to the rotation axis L



$$\vec{\tau} = \vec{\tau}_{||} + \vec{\tau}_{\perp}$$
$$\vec{\tau}_{||} = \hat{n} \cdot (\vec{r} \times \vec{F})$$

$\vec{\tau}_{\perp}$ has no effect if the system is only allowed to rotate about a given axis (z).

Triple Vector Product

$$\vec{A} \times (\vec{B} \times \vec{C}) =?$$

Let

$$\vec{D} = \vec{B} \times \vec{C} \quad \& \quad \vec{E} = \vec{A} \times \vec{D} = \vec{A} \times (\vec{B} \times \vec{C})$$

Then

$$\begin{aligned}
\vec{D} &= \hat{x}(B_yC_z - B_zC_y) + \hat{y}(B_zC_x - B_xC_z) + \\
&\quad \hat{z}(B_xC_y - B_yC_x) \\
\vec{E} &= \hat{x}(A_yD_z - A_zD_y) + \hat{y}(A_zD_x - A_xD_z) + \\
&\quad \hat{z}(A_xD_y - A_yD_x) \\
E_x &= A_yD_z - A_zD_y \\
&= A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z) \\
&= A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z \\
&= B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z) \\
&= B_x(A_xC_x + A_yC_y + A_zC_z) - \\
&\quad C_x(A_xB_x + A_yB_y + A_zB_z)
\end{aligned}$$

$$E_x = B_x(\vec{A} \cdot \vec{C}) - C_x(\vec{A} \cdot \vec{B})$$

Triple Vector Product (cont.)

$$E_x = B_x(\vec{A} \cdot \vec{C}) - C_x(\vec{A} \cdot \vec{B})$$

Similarly

$$E_y = B_y(\vec{A} \cdot \vec{C}) - C_y(\vec{A} \cdot \vec{B})$$

$$E_z = B_z(\vec{A} \cdot \vec{C}) - C_z(\vec{A} \cdot \vec{B})$$

$$\begin{aligned} \vec{E} &= E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \\ &= (\vec{A} \cdot \vec{C})(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) - \\ &\quad (\vec{A} \cdot \vec{B})(C_x\hat{x} + C_y\hat{y} + C_z\hat{z}) \\ &= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \end{aligned}$$

$$\boxed{\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}}$$

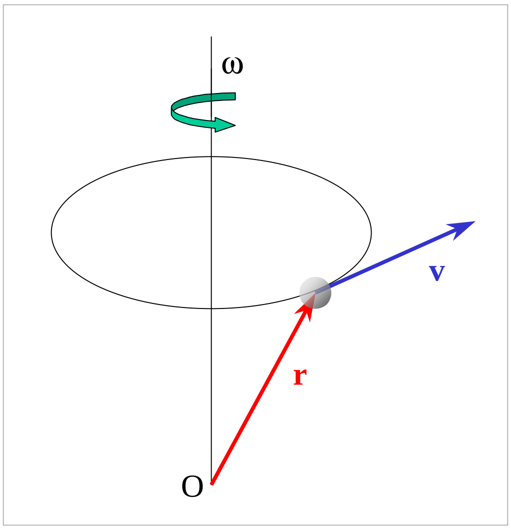
$$(\vec{B} \times \vec{C}) \times \vec{A} = (\vec{A} \cdot \vec{B})\vec{C} - (\vec{A} \cdot \vec{C})\vec{B}$$

$$\boxed{(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}}$$

Application of Triple Vector Product

Angular momentum:

$$\vec{L} = \vec{r} \times (m\vec{v})$$



Since

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\boxed{\vec{L} = m\vec{r} \times (\vec{\omega} \times \vec{r})}$$

Acceleration of a point in an object rotating with constant angular speed

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{v}$$

$$\boxed{\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})}$$