PC1134 Lecture 14

Topic

Differentiation of vectors

Objectives

To know how to differentiate a vector, when the vector is given in Cartesian components (unit vectors do not change) or in components in polar coordinates (unit vectors also change), and to be able to extend the procedure to other coordinate systems.

Relevance

Many physical quantities are vectors and they can depend on other variables. They may be functions of time.

Differentiation of Vector

Object moving in x-direction

$$x(t) \qquad \vec{r}(t) = x(t)\hat{x}$$

$$v(t) = \frac{dx}{dt} \qquad \vec{v}(t) = v(t)\hat{x} = \frac{dx}{dt}\hat{x}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \qquad \vec{a}(t) = a(t)\hat{x} = \frac{d^2x}{dt^2}\hat{x}$$

Object moving in y-direction

$$y(t) \qquad \vec{r}(t) = y(t)\hat{y}$$

$$v(t) = \frac{dy}{dt} \qquad \vec{v}(t) = v(t)\hat{y} = \frac{dy}{dt}\hat{y}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2} \qquad \vec{a}(t) = a(t)\hat{y} = \frac{d^2y}{dt^2}\hat{y}$$

Differentiation of Vector

General case

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$$

$$\vec{v}(t) = v_x(t)\hat{x} + v_y(t)\hat{y} + v_z(t)\hat{z}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$\vec{a}(t) = a_x(t)\hat{x} + a_y(t)\hat{y} + a_z(t)\hat{z}$$

$$\frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{x} + \frac{dv_y}{dt}\hat{y} + \frac{dv_z}{dt}\hat{z}$$

$$\frac{d^2\vec{r}}{dt} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z}$$

Any vector

$$\vec{A}(t) = A_x(t)\hat{x} + A_y(t)\hat{y} + A_z(t)\hat{z}$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{x} + \frac{dA_y}{dt}\hat{y} + \frac{dA_z}{dt}\hat{z}$$

$\frac{d\vec{A}}{dt}$ is a vector!

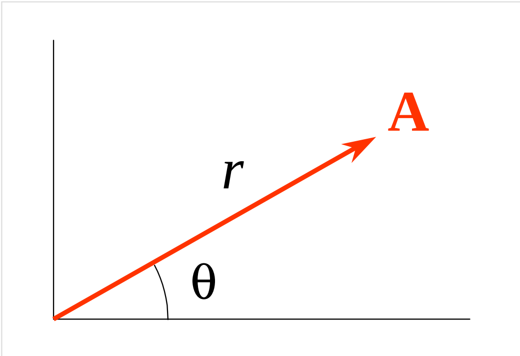
Differentiation of Vector

$$\frac{d}{dt}(a\vec{A}) = \frac{da}{dt}\vec{A} + a\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{B}$$

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$$

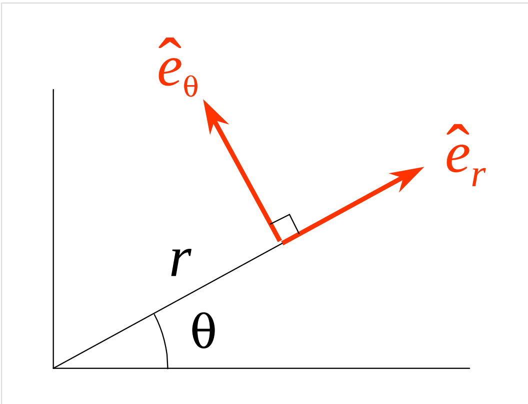
Polar Coordinates



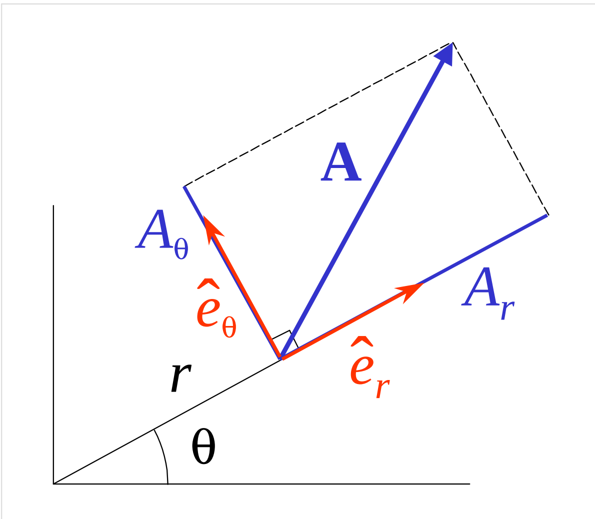
$$(x, y) \to (r, \theta)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$



$$\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta$$



Polar Coordinates

$$\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta$$

$$\frac{d\vec{A}}{dt} = ?$$

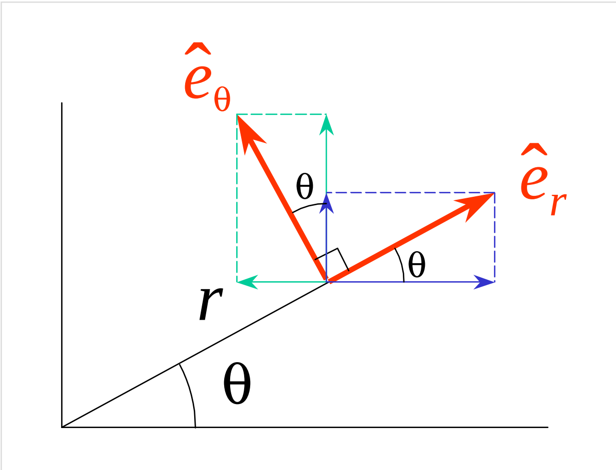
$\hat{e}_r$ and $\hat{e}_\theta$ are not constant vectors!

$$\frac{d\hat{e}_r}{dt} \neq 0, \quad \frac{d\hat{e}_\theta}{dt} \neq 0$$

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{d}{dt}\left(A_r \hat{e}_r + A_\theta \hat{e}_\theta\right) \\ &= A_r \frac{d\hat{e}_r}{dt} + \frac{dA_r}{dt}\hat{e}_r + A_\theta \frac{d\hat{e}_\theta}{dt} + \frac{dA_\theta}{dt}\hat{e}_\theta \end{aligned}$$

Polar Coordinates

$$\hat{e}_r = \cos\theta\hat{x} + \sin\theta\hat{y}, \quad \hat{e}_\theta = -\sin\theta\hat{x} + \cos\theta\hat{y}$$



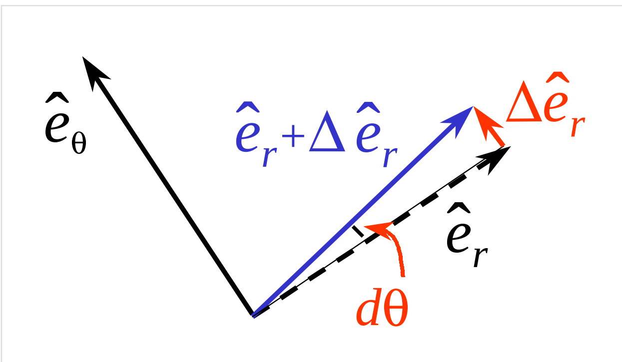
$$\begin{aligned} \frac{d\hat{e}_r}{dt} &= \frac{d}{dt}(\cos\theta\hat{x} + \sin\theta\hat{y}) \\ &= -\sin\theta\frac{d\theta}{dt}\hat{x} + \cos\theta\frac{d\theta}{dt}\hat{y} \\ &= (-\sin\theta\hat{x} + \cos\theta\hat{y})\frac{d\theta}{dt} \end{aligned}$$

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta\frac{d\theta}{dt}$$

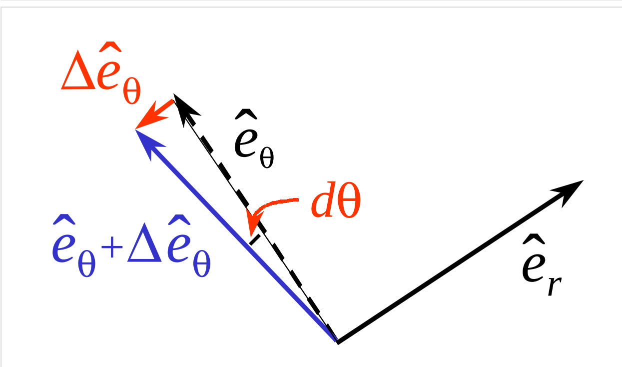
Polar Coordinates

$$\frac{d\hat{e}_\theta}{dt} = \frac{d}{dt}(-\sin\theta\hat{x} + \cos\theta\hat{y})$$
$$= -\cos\theta\frac{d\theta}{dt}\hat{x} - \sin\theta\frac{d\theta}{dt}\hat{y}$$
$$= (-\cos\theta\hat{x} - \sin\theta\hat{y})\frac{d\theta}{dt}$$

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\frac{d\theta}{dt}$$



$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta\frac{d\theta}{dt}$$



$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\frac{d\theta}{dt}$$

Polar Coordinates

$$\frac{d\vec{A}}{dt} = A_r \frac{d\hat{e}_r}{dt} + \frac{dA_r}{dt}\hat{e}_r + A_\theta \frac{d\hat{e}_\theta}{dt} + \frac{dA_\theta}{dt}\hat{e}_\theta$$

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \frac{d\theta}{dt}$$

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \frac{d\theta}{dt}$$

$$\frac{d\vec{A}}{dt} = A_r \frac{d\theta}{dt}\hat{e}_\theta + \frac{dA_r}{dt}\hat{e}_r - A_\theta \frac{d\theta}{dt}\hat{e}_r + \frac{dA_\theta}{dt}\hat{e}_\theta$$

$$\frac{d\vec{A}}{dt} = \left(\frac{dA_r}{dt} - A_\theta \frac{d\theta}{dt}\right)\hat{e}_r + \left(\frac{dA_\theta}{dt} + A_r \frac{d\theta}{dt}\right)\hat{e}_\theta$$