PC1134 Lecture 15

Topic

Field, Potential & Gradient

Objectives

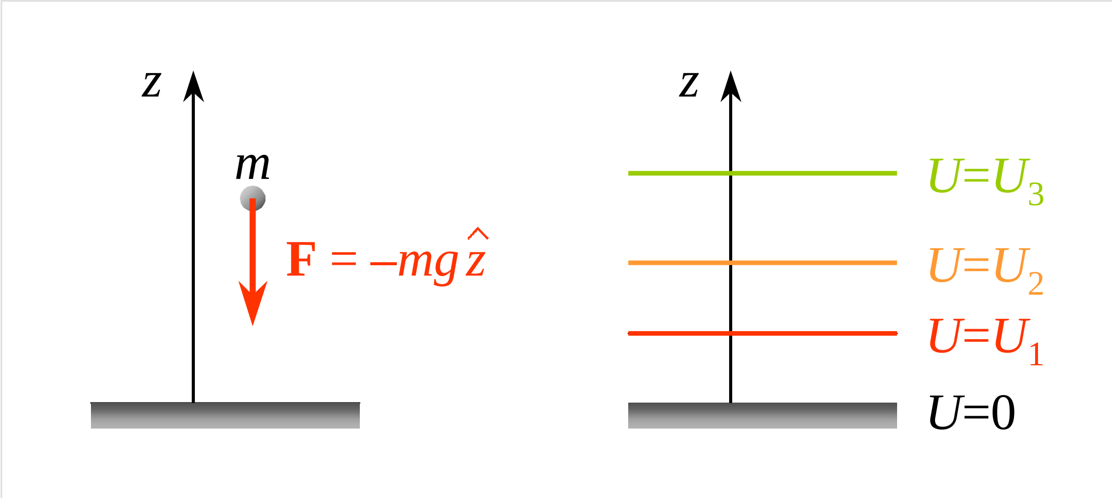
To understand the concept of gradient, its geometrical and physical meanings; and to be able to apply it to force/field and potential.

Relevance

- Field and potential
- Gradient, rate of change of a function along any direction
- Differential relationship between force/field and potential

Gravitational Force
(near the surface of Earth)

$$\vec{F} = -mg\hat{z}$$



Gravitational field:

$$\vec{g} = -g\hat{z}$$

Potential:

$$gz = U(z)$$

Potential energy:

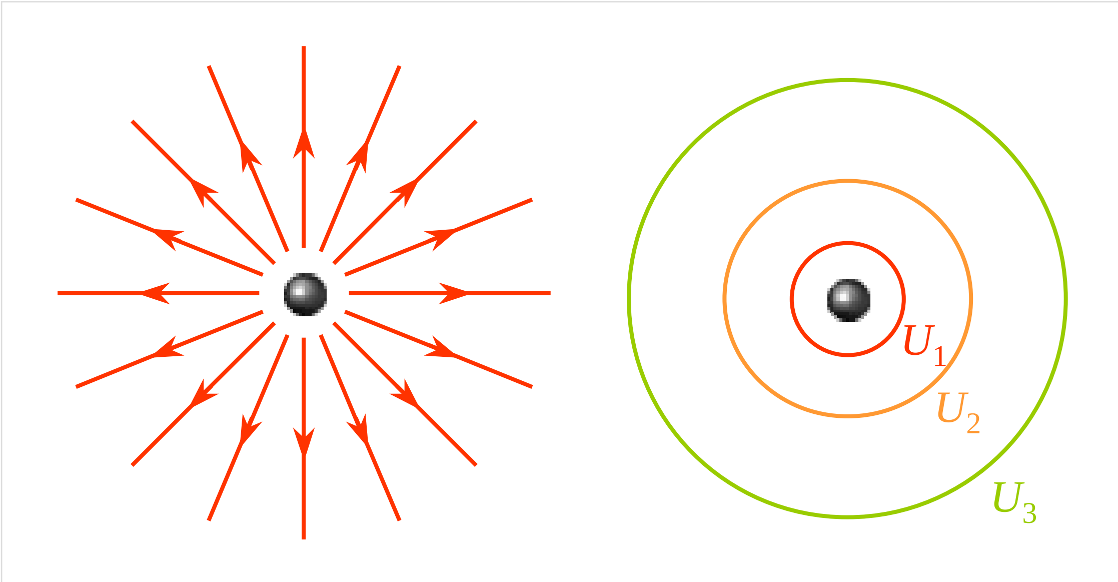
$$mgz$$

Equipotential surface:

$$U(z) = gz = \text{const}$$

$$\Longrightarrow \quad z = \text{const.}$$

Field & Potential of a Point Charge



Electrostatic force between two point charges:

$$\vec{F} = \frac{kq_1q_2}{r^2}\hat{r}$$

Electric field:

$$\vec{E} = \frac{kq}{r^2}\hat{r}$$

Potential:

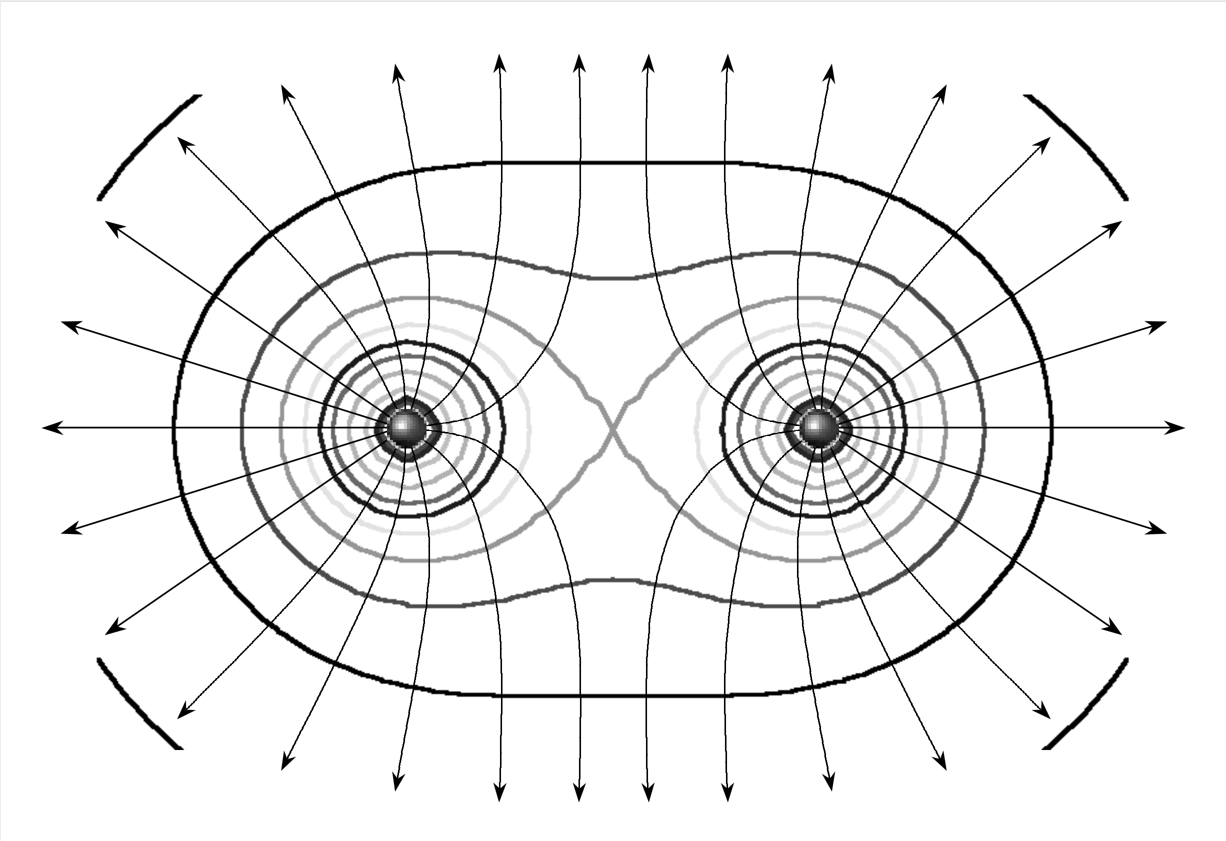
$$V(r) = \frac{kq}{r}$$

Equipotential surface:

$$V(r) = \frac{kq}{r} = \text{const.}$$

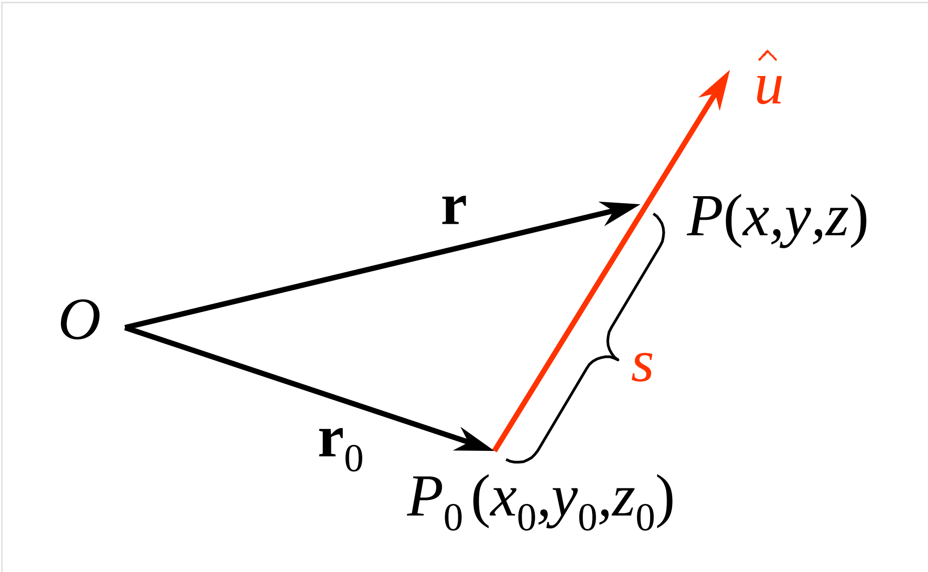
$$\Longrightarrow \quad r = \text{const.}$$

Potential of Two Charges



How does the potential change along any direction?

Potential



Potential at P_0: $\phi(x_0, y_0, z_0)$

Potential at P: $\phi(x, y, z)$

Let

$$\vec{r} - \vec{r}_0 = s\hat{u}$$

$$\Delta\phi = \phi(x, y, z) - \phi(x_0, y_0, z_0)$$

For a small change in position along the direction of $\hat{u}$

$$\Delta\phi \approx \frac{d\phi}{ds} ds$$

$$\frac{d\phi}{ds} = ?$$

Potential

Let

$$\hat{u} = a\hat{x} + b\hat{y} + c\hat{z}$$

$$a^2 + b^2 + c^2 = 1$$

$$\Delta\vec{r} = \vec{r} - \vec{r}_0 = s\hat{u} \quad \Longrightarrow \quad \vec{r} = \vec{r}_0 + s\hat{u}$$

$$\begin{cases} x = x_0 + as \\ y = y_0 + bs \\ z = z_0 + cs \end{cases}$$

$$\phi(x, y, z) = \phi[x(s), y(s), z(s)]$$

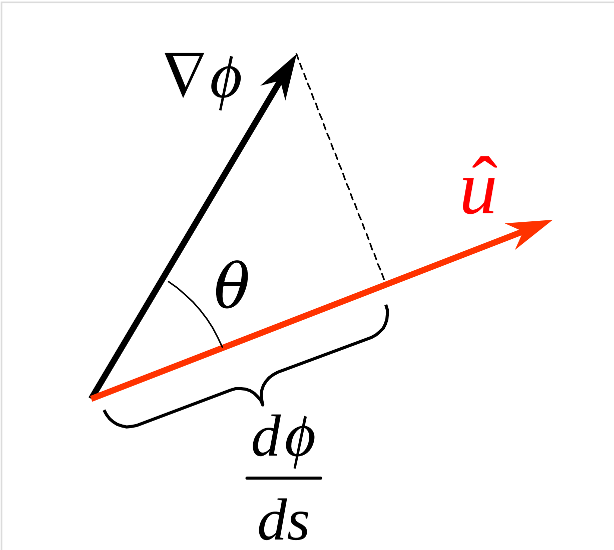
$$\begin{aligned} \frac{d\phi}{ds} &= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} \\ &= a\frac{\partial\phi}{\partial x} + b\frac{\partial\phi}{\partial y} + c\frac{\partial\phi}{\partial z} \\ &= (a\hat{x} + b\hat{y} + c\hat{z}) \cdot \left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right) \end{aligned}$$

$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{u}$$

$$\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z} = \mathrm{grad}\phi$$

What is $\nabla\phi$?

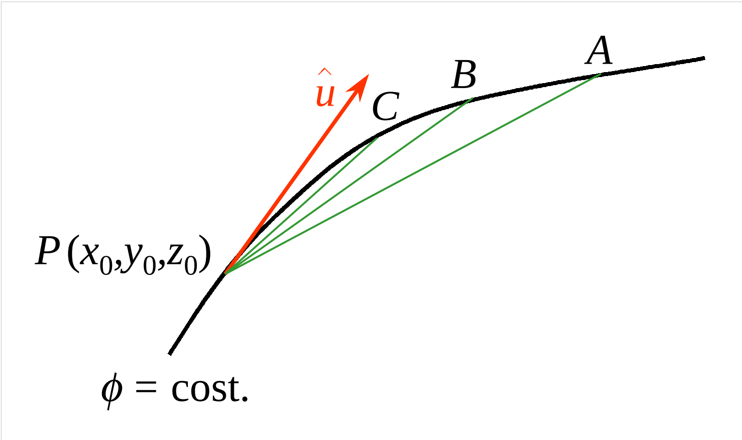
$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{u} = |\nabla\phi| \cos\theta$$



The largest value of $d\phi/ds$ is found in the direction of $\nabla\phi$!

What is $\nabla\phi$?

Consider an equipotential surface:



$$\phi_P = \phi_A = \phi_B = \phi_C$$

$$\Delta\phi = 0 \quad \rightarrow \quad \frac{\Delta\phi}{\Delta s} = 0$$

When

$$\Delta s \rightarrow 0$$

$$PA \rightarrow PB \rightarrow PC \rightarrow \hat{u}$$

$$\frac{\Delta\phi}{\Delta s} \rightarrow \frac{d\phi}{ds} = 0 \;\; \text{along } \hat{u}$$

.

$$\nabla\phi \cdot \hat{u} = |\nabla\phi|\cos\theta = 0 \quad \Longrightarrow \quad \nabla\phi \perp \hat{u}$$

$\nabla\phi$ is perpendicular to the surface $\phi =$ constant.

Gravitational Potential

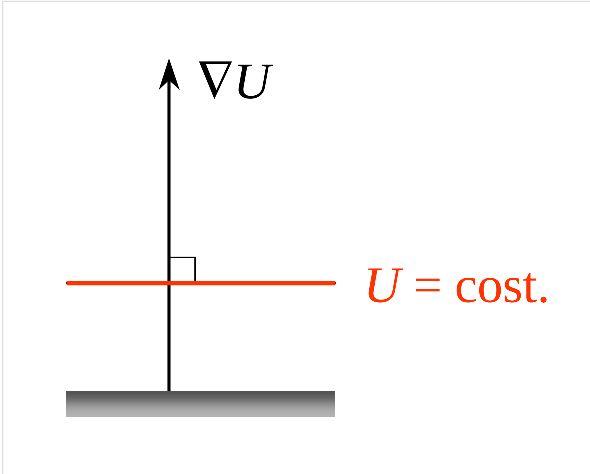
$$U(z) = gz$$

$$\nabla U = g\hat{z}$$

Equipotential:

$$U(z) = gz = \text{const.}$$

$$z = \text{const.}$$



An equipotential surface (U = constant) is a plane parallel to the x-y plane.

dU/ds is the largest in the direction of $\hat{z}$.

$$\nabla U = g\hat{z} = -\vec{g}$$

$$\vec{g} = -\nabla U$$

$$\vec{F} = m\vec{g}$$

$\nabla\phi$ in Different Coordinate Systems

∇U in Cartesian coordinates

$$\nabla U = \hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}$$

∇U in polar coordinates

$$\nabla U = \hat{e}_r\frac{\partial U}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial U}{\partial \theta}$$

∇U in cylindrical coordinates

$$\nabla U = \hat{e}_r\frac{\partial U}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial U}{\partial \theta} + \hat{e}_z\frac{\partial U}{\partial z}$$

∇U in spherical coordinates

$$\nabla U = \hat{e}_r\frac{\partial U}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial U}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

Point Charge

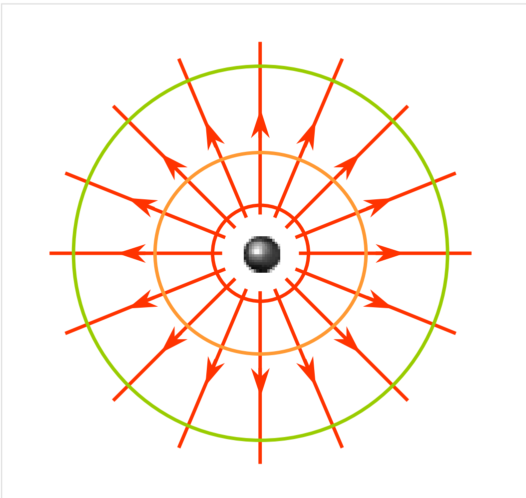
Electrostatic potential:

$$V(r) = \frac{kq}{r}$$

$$\nabla V = \hat{e}_r kq \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{kq}{r^2}\hat{e}_r$$

Equipotential surface:

$$r = \text{const.}$$



$$\nabla V = -\vec{E}$$

$$\vec{F} = q_2 \vec{E}$$