PC1134 Lecture 16

Topic

The ∇ operator, divergence and curl

Objectives

To understand the properties of the operator ∇.
To know how to calculate divergence, curl and Laplacian in Cartesian coordinates.

Relevance

- Field and potential: potential (scalar) is easier to handle in many cases

$$\vec{E} = -\nabla \phi$$

- Being a derivative in nature, ∇ appears in many differential equations that describing physical systems and processes.

The ∇ Operator

$$\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}_{\nabla\ \text{(operator)}}\phi$$

$$\boxed{\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$

∇ has no meaning by itself. It must act on a function.

Divergence

Operate ∇ on a vector?

$$\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$$

$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \equiv \text{div}\vec{V}$$

$\nabla \cdot \vec{V}$ is the divergence of $\vec{V}$

Physically, divergence of a vector represents a source that produces divergent field lines. For example

$$\nabla \cdot \vec{E} = 4\pi\rho$$

Curl

$$\begin{aligned}
\nabla \times \vec{V} &= \left| \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{array} \right| \\
&= \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \\
&\quad \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \\
&\equiv \mathrm{curl}\vec{V}
\end{aligned}$$

$\nabla \times \vec{V}$ is the curl of $\vec{V}$

The curl of a vector represents some source that produces looped field lines or current flow. For example

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J}$$

Laplacian Operator

$\nabla\phi$ is a vector. $\nabla\cdot\nabla\phi = ?$

$$\begin{aligned}
\nabla\cdot\nabla\phi &= \text{div grad}\phi \\
&= \nabla\cdot\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\phi \\
&= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) \\
&= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}
\end{aligned}$$

$$\nabla\cdot\nabla\phi = \underbrace{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)}_{\nabla^2\ \text{(operator)}}\phi$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}}$$

∇^2 is the Laplacian operator.

Examples of Laplacian Operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplace's equation

$$\nabla^2 \phi = 0$$

Wave equation

$$\nabla^2 \phi = \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2}$$

Diffusion equation

$$\nabla^2 \phi = \frac{1}{a^2} \frac{\partial \phi}{\partial t}$$

Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 \phi + V\phi = E\phi$$

Vector Operations (Reference)

Cartesian

$$\nabla\psi = \hat{x}\frac{\partial\psi}{\partial x} + \hat{y}\frac{\partial\psi}{\partial y} + \hat{z}\frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

Cylindrical

$$\nabla\psi = \hat{e}_\rho\frac{\partial\psi}{\partial\rho} + \hat{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{z}\frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A} = \hat{e}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{e}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical

$$\nabla\psi = \hat{e}_r\frac{\partial\psi}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\vec{A} = \hat{e}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{e}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$
$$+\hat{e}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Vector Identities (Reference)

$$\nabla \cdot \nabla\phi = \text{div grad}\phi = \nabla^2\phi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla \times \nabla\phi = \text{curl grad}\phi = 0$$

$$\nabla(\nabla\cdot\vec{V}) = \text{grad div}\vec{V} = \hat{x}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x\partial y} + \frac{\partial^2 V_z}{\partial x\partial z}\right) + \hat{y}\left(\frac{\partial^2 V_x}{\partial x\partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y\partial z}\right) + \hat{z}\left(\frac{\partial^2 V_x}{\partial x\partial z} + \frac{\partial^2 V_y}{\partial y\partial z} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

$$\nabla \cdot (\nabla \times \vec{V}) = \text{div curl } \vec{V} = 0$$

$$\nabla \times (\nabla \times \vec{V}) = \text{curl curl } \vec{V} = \nabla(\nabla \cdot \vec{V}) - \nabla^2\vec{V}$$

$$\nabla \cdot (\phi\vec{V}) = \phi(\nabla \cdot \vec{V}) + \vec{V} \cdot (\nabla\phi)$$

$$\nabla \times (\phi\vec{V}) = \phi(\nabla \times \vec{V}) - \vec{V} \times (\nabla\phi)$$

$$\nabla \cdot (\vec{U} \times \vec{V}) = \vec{V} \cdot (\nabla \times \vec{U}) - \vec{U} \cdot (\nabla \times \vec{V})$$

$$\nabla \times (\vec{U} \times \vec{V}) = (\vec{V} \cdot \nabla)\vec{U} - (\vec{U} \cdot \nabla)\vec{V} - \vec{V}(\nabla \cdot \vec{U}) + \vec{U}(\nabla \cdot \vec{V})$$

$$\nabla(\vec{U} \cdot \vec{V}) = \vec{U} \times (\nabla \times \vec{V}) + (\vec{U} \cdot \nabla)\vec{V} + \vec{V} \times (\nabla \times \vec{U}) + (\vec{V} \cdot \nabla)\vec{U}$$

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = 0$$