PC1134 Lecture 17

Topic

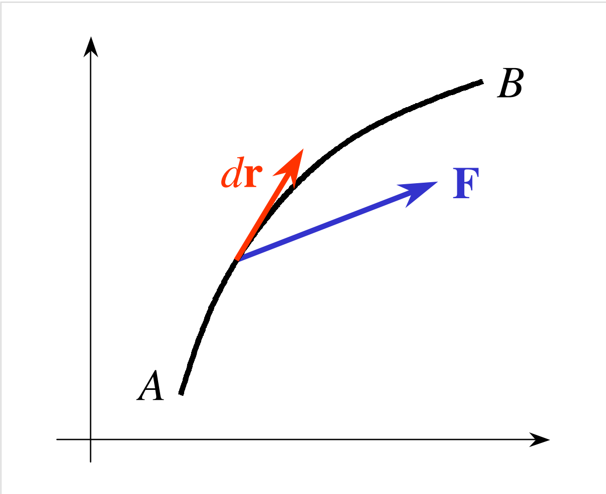
Line integrals

Application example

Work done by a force.

$$dW = \vec{F} \cdot d\vec{r}$$

$$W = \int_A^B \vec{F} \cdot d\vec{r}$$



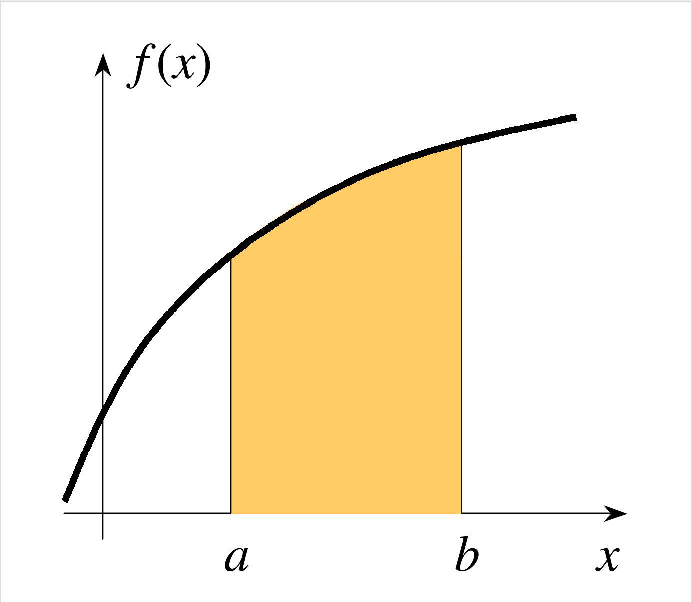
Scope

- Review of integration
- Line integral
- Calculation of line integral

Integration

The difinite integral of $f(x)$ between the lower limit $x = a$ and the upper limit $x = b$. $f(x)$ is called the integrand.

$$I = \int_a^b f(x)dx$$

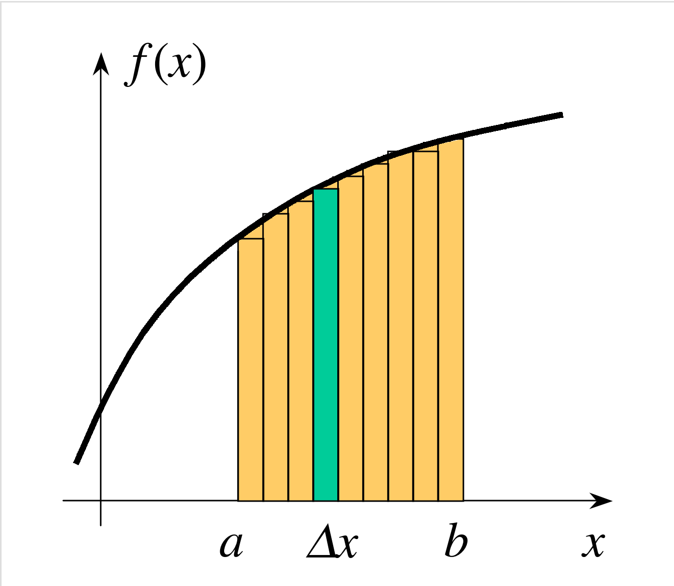


I is the area under the curve.

Integration from First Principles

Divide the area into slices of width dx (not required to be uniform). The area of each slide is

$$dA \approx f(x)dx$$



Total area

$$A \approx \sum f(x)\Delta x$$

The error is reduced when Δx is small. In the limit of $\Delta x \to 0$,

$$A = \lim_{\Delta x \to 0} \sum f(x)dx = \int_a^b f(x)dx$$

Properties of Integration

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Integration of Simple Functions

$$
\begin{aligned}
\int a\,dx &= ax \\
\int x^n\,dx &= \frac{1}{n+1}x^{n+1} \\
\int e^{ax}\,dx &= \frac{e^{ax}}{a} \\
\int \frac{dx}{x} &= \ln x \\
\int \cos ax\,dx &= \frac{\sin ax}{a} \\
\int \sin ax\,dx &= -\frac{\cos ax}{a} \\
\int \frac{dx}{a^2+x^2}dx &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \\
\int \frac{dx}{\sqrt{a^2-x^2}} &= \sin^{-1}\left(\frac{x}{a}\right)
\end{aligned}
$$

......

Integration by Substitution

Example 1

$$I = \int \frac{dx}{\sqrt{1 - x^2}}$$

Let

$$x = \sin u$$

then

$$dx = \cos u du$$

$$I = \int \frac{\cos u du}{\sqrt{1 - \sin^2 u}} = \int \frac{\cos u du}{\sqrt{\cos^2 u}} = \int du = u$$

$$I = \sin^{-1} x$$

Example 2

$$I = \int \frac{dx}{x^2 + 4x + 7} = \int \frac{dx}{(x+2)^2 + 3}$$

Let $y = x + 2$, then $dy = dx$ and

$$I = \int \frac{dy}{y^2 + 3} = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{y}{\sqrt{3}} \right) = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x+2}{\sqrt{3}} \right)$$

Integration by Parts

If u and v are functions of x

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$$

Integrate both sides

$$uv = \int u\frac{dv}{dx}dx + \int \frac{du}{dx}vdx$$

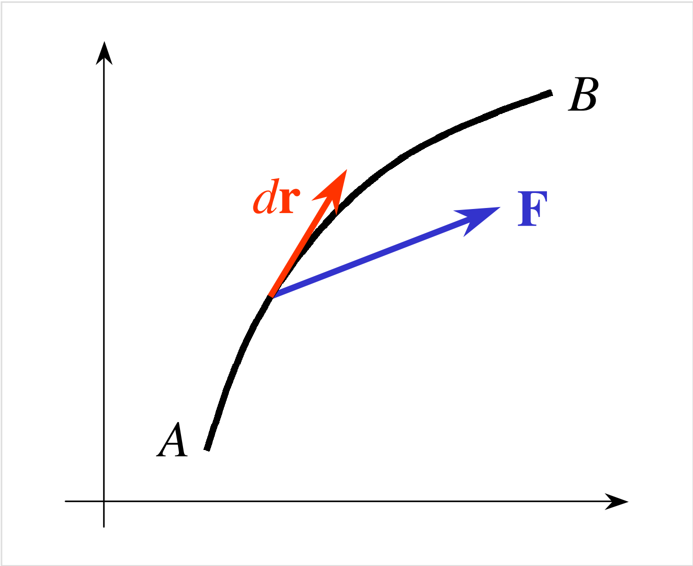
This can be rewritten as

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}vdx$$

Example

$$I = \int x \sin x dx = \int x\frac{d(-\cos x)}{dx}dx = x(-\cos x)$$
$$-\int (1)(-\cos x)dx = -x\cos x + \sin x$$

Work Done by a Varying Force



$$dW = \vec{F} \cdot d\vec{r}$$

$$W = \int \vec{F} \cdot d\vec{r}$$

- $\vec{F}$ is a function of position

$$\vec{F}(\vec{r}) \text{ or } \vec{F}(x, y, z)$$

- Object is restricted to move along the curved path.

Line Integral

$$W = \int \vec{F} \cdot d\vec{r}$$

2D

A curve is described by one equation

$$\phi(x, y) = 0 \quad \Longrightarrow \quad y = \psi(x)$$

$$\int \vec{F}(x, y) \cdot d\vec{r} \;\Longrightarrow\; \int f(x)dx$$

or

$$\int \vec{F}(x, y) \cdot d\vec{r} \;\Longrightarrow\; \int f(s)ds$$

3D

Two equations relating x, y and z are required to describe a curve

$\Longrightarrow$ Only one independent variable

$\Longrightarrow$ One dimensional integral

Example

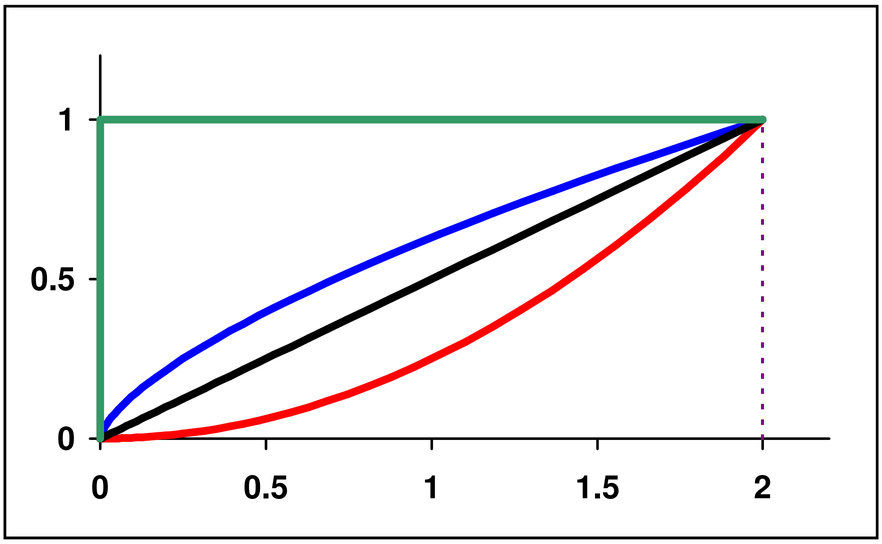
$$\vec{F} = xy\hat{x} - y^2\hat{y}$$

Calcualte

$$W = \int \vec{F} \cdot d\vec{r}$$

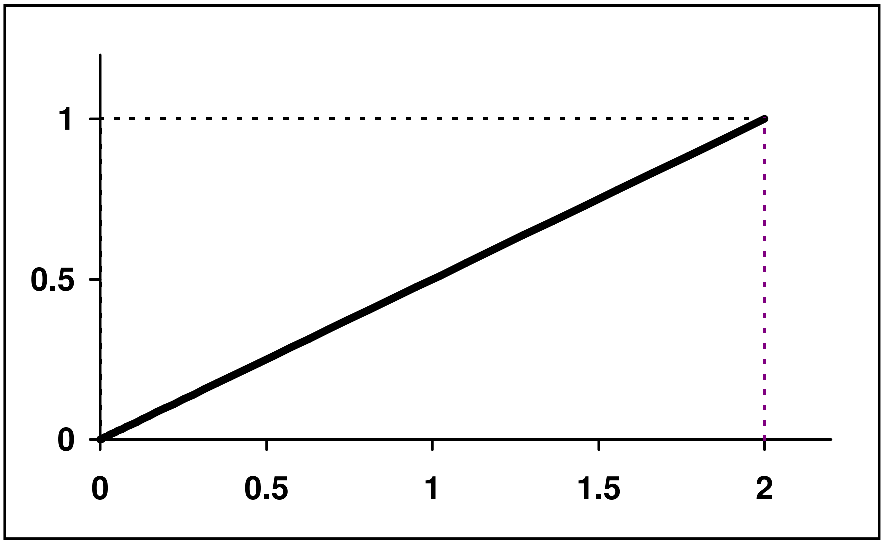
along

1. Path 1 (black) $y = x/2$
2. Path 2 (red) $y = x^2/4$
3. Path 3 (green) $x = 0$ and $y = 1$
4. Path 4 (blue) $x = 2t^3$, $y = t^2$.



Example

Path 1 (black)



$$y = \frac{1}{2}x, \qquad dy = \frac{1}{2}dx$$

$$d\vec{r} = dx\hat{x} + dy\hat{y} = dx\hat{x} + \frac{1}{2}dx\hat{y} = \left(\hat{x} + \frac{1}{2}\hat{y}\right) dx$$

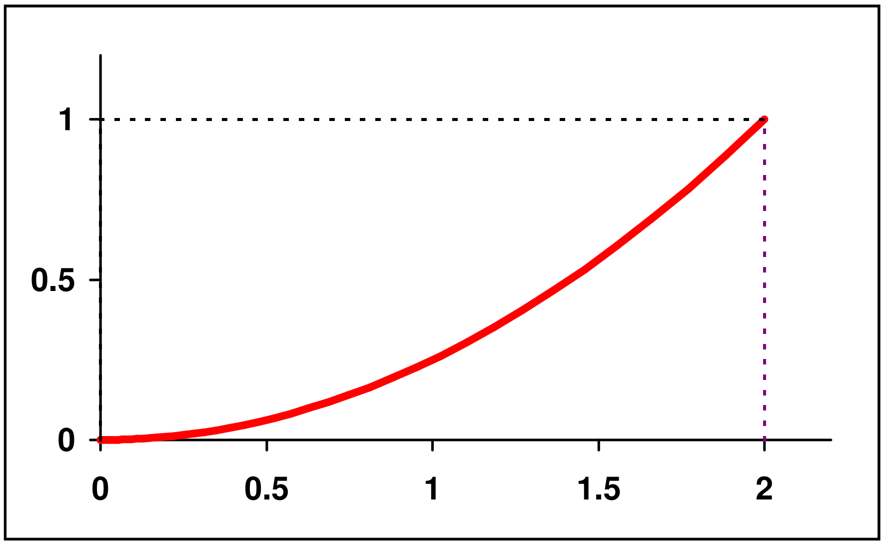
$$\vec{F}\cdot d\vec{r} = F_x dx + F_y dy = xydx - y^2\frac{1}{2}dx = \left(xy - \frac{1}{2}y^2\right) dx$$

$$= \left(\frac{1}{2}x^2 - \frac{1}{8}x^2\right) dx = \frac{3}{8}x^2 dx$$

$$W_1 = \int \vec{F} \cdot d\vec{r} = \int_0^2 \frac{3}{8}x^2 dx = \left.\frac{x^3}{8}\right|_0^2 = 1$$

Example

Path 2 (red)



$$y = \frac{1}{4}x^2, \qquad dy = \frac{1}{2}x\,dx$$

$$d\vec{r} = dx\hat{x} + dy\hat{y} = dx\hat{x} + \frac{1}{2}x\,dx\hat{y} = \left(\hat{x} + \frac{1}{2}x\hat{y}\right) dx$$

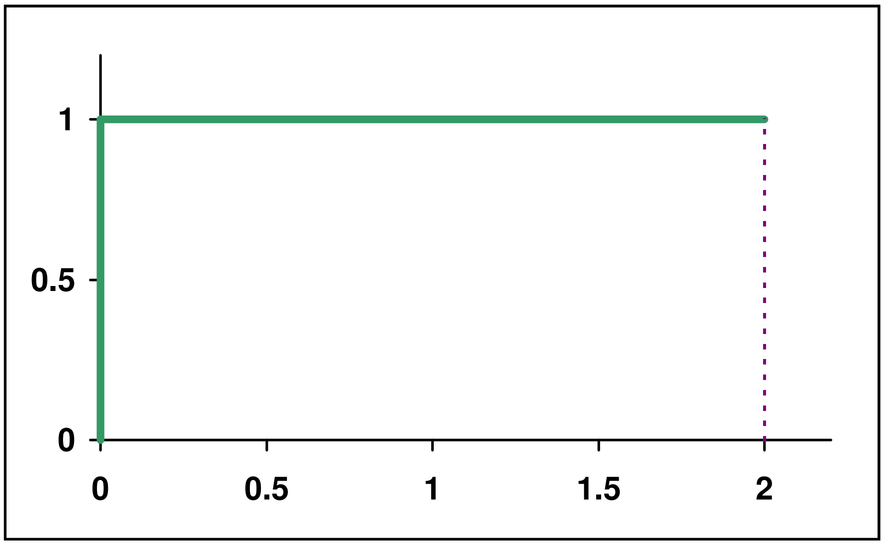
$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy$$

$$= \left(xy - \frac{1}{2}xy^2\right) dx = \left(\frac{1}{4}x^3 - \frac{1}{32}x^5\right) dx$$

$$W_2 = \int \vec{F} \cdot d\vec{r} = \int_0^2 \left(\frac{1}{4}x^3 - \frac{1}{32}x^5\right) dx = \frac{2}{3}$$

Example

Path 3 (green)



Along the vertical segment, $x = 0$ and $dx = 0$

$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy = -y^2 dy$$

$$W_{3a} = \int \vec{F} \cdot d\vec{r} = \int_0^1 (-y^2) dy = -\frac{1}{3}$$

Along the horizontal segment, $y = 1$ and $dy = 0$

$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy = x dx$$

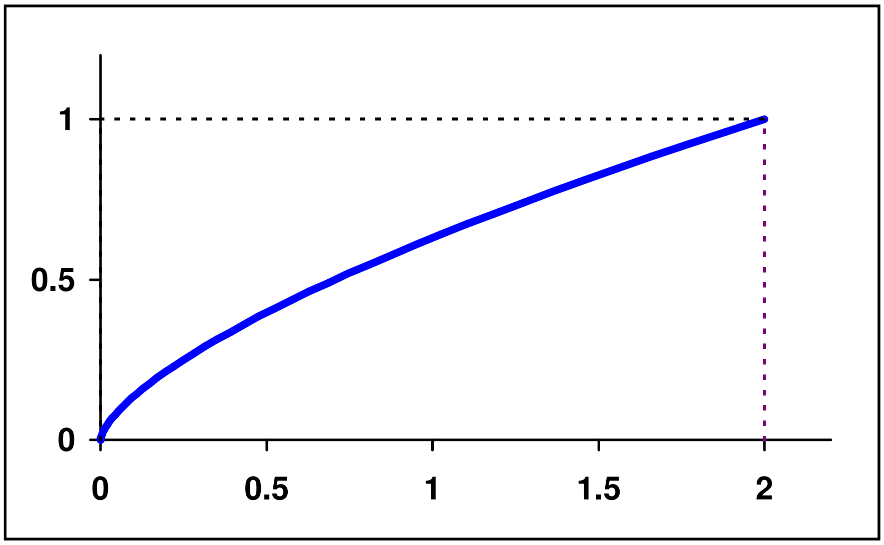
$$W_{3b} = \int \vec{F} \cdot d\vec{r} = \int_0^2 x dx = 2$$

$$W_3 = W_{3a} + W_{3b} = -\frac{1}{3} + 2 = \frac{5}{3}$$

Example

Path 4 (blue):

$$x = 2t^3, \qquad y = t^2$$



$$\vec{F} = xy\hat{x} - y^2\hat{y} = 2t^5\hat{x} - t^4\hat{y}$$

$$d\vec{r} = dx\hat{x} + dy\hat{y} = 6t^2dt\hat{x} + 2tdt\hat{y}$$

$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy = 12t^7dt - 2t^5dt$$

$$W_4 = \int \vec{F} \cdot d\vec{r} = 12\int_0^1 t^7dt - 2\int t^5dt = \frac{7}{6}$$

Summary

- $\vec{F}(x, y, z)$ is a function of one variable because x, y and z are related.

- This variable can be one of x, y and z or can be something else.

- The integration over $\vec{F} \cdot d\vec{r}$ is an one-dimensional integral over this variable.

- The line integral between two points can be path dependent. ($W_1 = 1$, $W_2 = 2/3$, $W_3 = 5/3$, $W_4 = 7/6$).