PC1134 Lecture 18

Topic

Conservative field and potential

Objectives

To understand the concept of conservative and nonconservative fields.

To be able to judge whether a given force field is conservative.

To be able to find the potential of a conservative field.

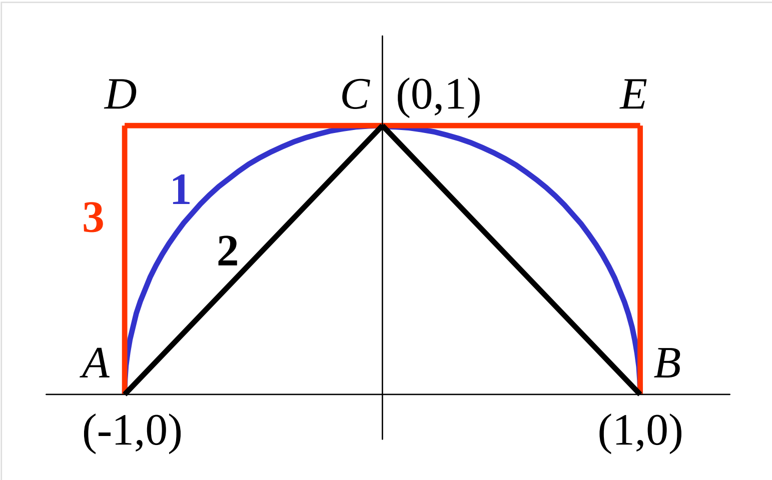
Example of Line Integral

Find

$$I = \int \frac{xdy - ydx}{x^2 + y^2}$$

along

1. the semicircle (1);
2. dotted lines (2);
3. path (3).



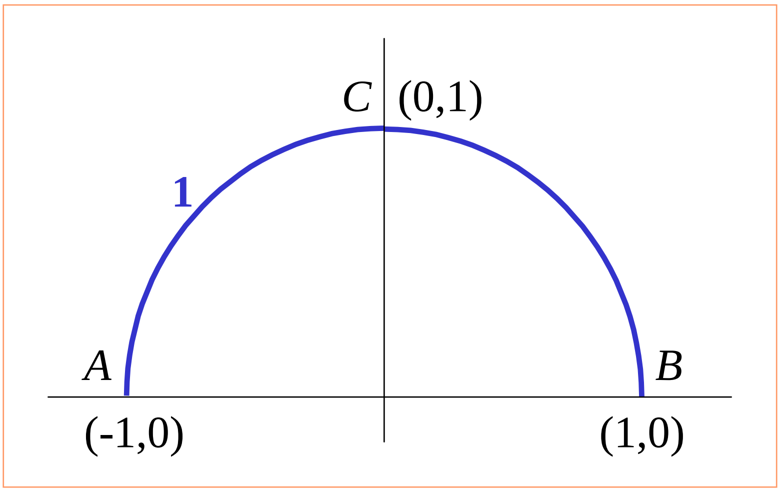
The above is

$$\int \vec{F} \cdot d\vec{r}, \qquad \vec{F} = \frac{-y\hat{x} + x\hat{y}}{x^2 + y^2}$$

Example of Line Integral

Path 1:

$$x^2 + y^2 = 1$$



Or

$$x = \cos\theta \quad dx = -\sin\theta d\theta$$

$$y = \sin\theta \quad dy = \cos\theta d\theta$$

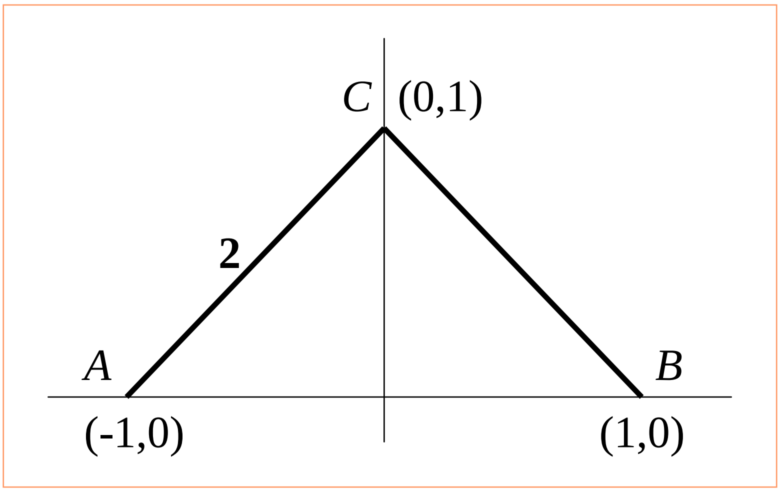
At point A: $\theta = \pi$
At point B: $\theta = 0$

$$\frac{xdy - ydx}{x^2 + y^2} = \frac{\cos^2\theta d\theta + \sin^2\theta d\theta}{1} = d\theta$$

$$I = \int_{\pi}^{0} d\theta = -\pi$$

Example of Line Integral

Path 2:



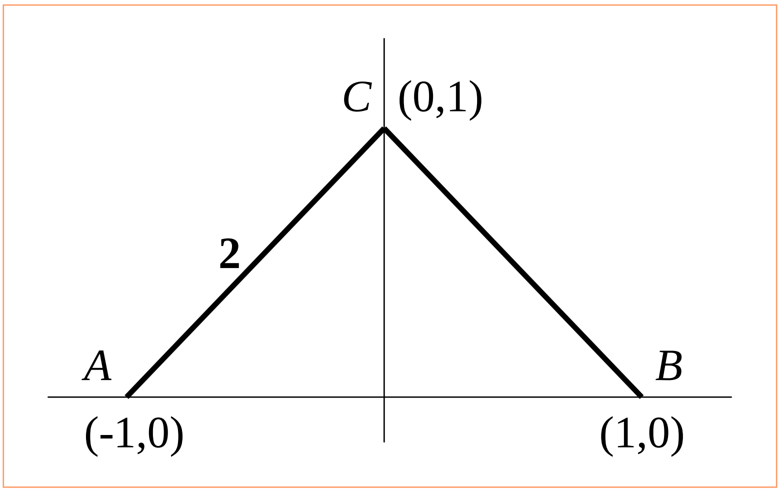
from A to C:

$$y = x + 1, \;\; dy = dx$$

$$\begin{aligned} I_{AC} &= \int_{-1}^{0} \frac{x dx - (x+1) dx}{x^2 + (x+1)^2} = \int_{-1}^{0} \frac{-dx}{2x^2 + 2x + 1} \\ &= \int_{-1}^{0} \frac{-2dx}{(2x+1)^2 + 1} = -\tan^{-1}(2x+1)\Big|_{-1}^{0} \\ &= -\tan^{-1} 1 + \tan^{-1}(-1) = -\frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = -\frac{\pi}{2} \end{aligned}$$

Example of Line Integral

Path 2:



from C to B:

$$y = 1 - x, \;\; dy = -dx$$

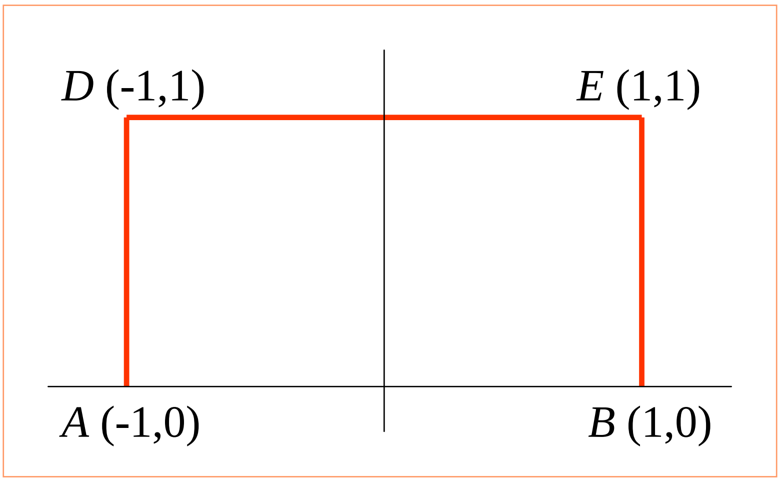
$$I_{CB} = \int_0^1 \frac{-xdx - (1-x)dx}{x^2 + (1-x)^2} = \int_0^1 \frac{-dx}{2x^2 - 2x + 1}$$

$$= \int_0^1 \frac{-2dx}{(2x-1)^2 + 1} = -\tan^{-1}(2x-1)\Big|_0^1 = -\frac{\pi}{2}$$

$$I = I_{AC} + I_{CB} = -\pi$$

Example of Line Integral

Path 3:



from A to D: $x = -1$, $dx = 0$

$$I_{AD} = \int_0^1 \frac{-dy}{y^2+1} = -\tan^{-1} y\Big|_0^1 = -\frac{\pi}{4}$$

from D to E: $y = 1$, $dy = 0$

$$I_{DE} = \int_{-1}^1 \frac{-dx}{x^2+1} = -\tan^{-1} x\Big|_{-1}^1 = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$$

from E to B: $x = 1$, $dx = 0$

$$I_{EB} = \int_1^0 \frac{dy}{y^2+1} = \tan^{-1} y\Big|_1^0 = -\frac{\pi}{4}$$

$$I = -\pi$$

Summary

- $\vec{F}(x, y, z)$ is a function of one variable because x, y and z are related.

- This variable can be one of x, y and z or can be something else.

- The integration over $\vec{F} \cdot d\vec{r}$ is an one-dimensional integral over this variable.

- The line integral between two points may or may not be path dependent.

- If work done depdends on path, the force field is non-conservative. If work done does not depend on path, the force field is conservative.

Conservative Field

$$\vec{F} = \frac{-y\hat{x} + x\hat{y}}{x^2 + y^2} = -\frac{y}{x^2 + y^2}\hat{x} + \frac{x}{x^2 + y^2}\hat{y}$$

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{(x^2 + y^2)} & \dfrac{x}{(x^2 + y^2)} & 0 \end{vmatrix} \\
&= \hat{z}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2}\right)\right] \\
&= \hat{z}\left[\frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2}\right. \\
&\qquad \left. + \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2}\right] \\
&= 0
\end{aligned}$$

Non-conservative Field

$$\vec{F} = xy\hat{x} - y^2\hat{y}$$

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -y^2 & 0 \end{vmatrix} \\
&= \hat{z}\left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xy)\right] \\
&= -x\hat{z}
\end{aligned}$$

$\nabla \times \vec{F} = 0 \quad \Longleftrightarrow \quad$ conservative

$\nabla \times \vec{F} \neq 0 \quad \Longleftrightarrow \quad$ non-conservative

Conservative Fields and Potentials

A vector field $\vec{F}$ that has continuous partial derivatives in a simply connected region R is conservative if, and only if, any of the following is true:

1. The integral $\int_A^B \vec{F} \cdot d\vec{r}$, where A and B line in the region R, is independent of the path from A to B. Hence the integral $\int_C \vec{F} \cdot d\vec{r}$ around any closed loop in R is zero.

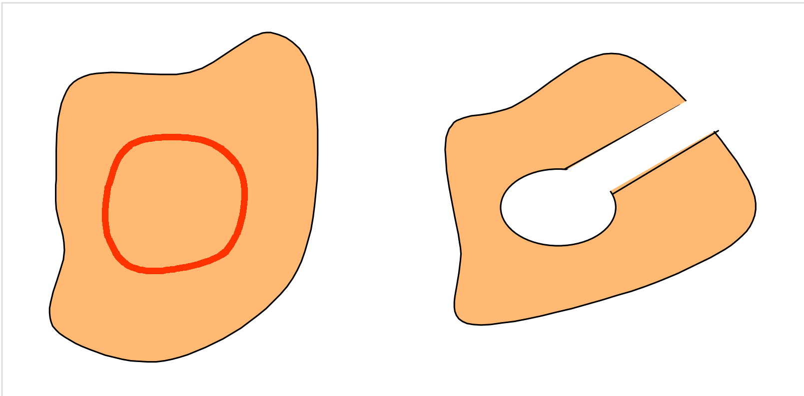
2. There exists a single-valued function W of position such that $\vec{F} = \nabla W$.

3. $\nabla \times \vec{F} = 0$.

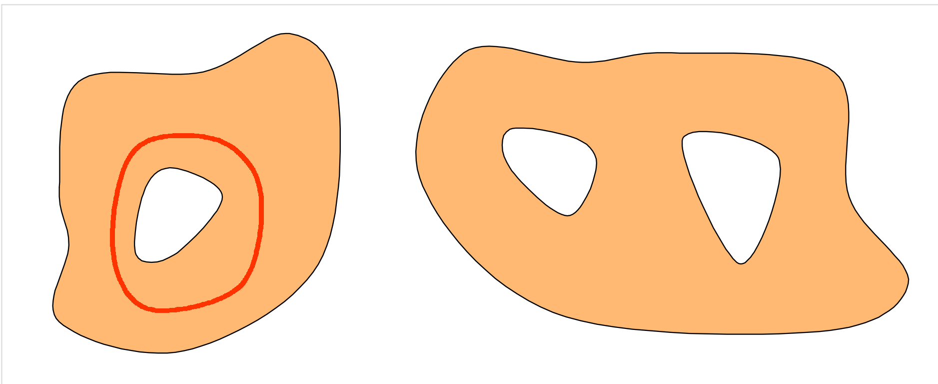
4. $\vec{F} \cdot d\vec{r}$ is an exact differential.

Connectivity of Regions

A region is *simply connected* if any simple closed curve in the region can be shrunk to a point without encouraging any points not in the region.



Simply connected regions



Not simply connected regions

Exact Differential

Differentials which integrate directly are called *exact differentials*, whereas those that do not are *inexact differentials*

$$df = xdy + ydx, \quad \Longrightarrow \quad f(x,y) = xy + c$$

$xdy + 3ydx$ cannot be integrated directly.

If

$$df = A(x,y)dx + B(x,y)dy$$

the necessary and the sufficient condition for a differential to be exact is

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.$$

This is because

$$f'_x = A(x,y) \text{ and } f'_y = B(x,y)$$

If we require $f''_{xy} = f''_{yx}$, the above equation must be satisfied.

Conservative Force Fields

- Assume the work done is independent of path taken, then the work must be a function only of the positions of the starting and ending points.

$$\int_A^B \vec{F} \cdot d\vec{r} = W(B) - W(A)$$

  W is a single-valued scalar function of position.

- If A and B are separated by an infinitesimal displacement $d\vec{r}$, then

$$\vec{F} \cdot d\vec{r} = dW$$

  i.e. $\vec{F} \cdot d\vec{r}$ is an exact differential.

- From $\dfrac{dW}{dr} = \nabla W \cdot \hat{u}$

$$\Longrightarrow dW = \frac{dW}{dr} dr = \nabla W \cdot \hat{u} dr = \nabla W \cdot d\vec{r}$$

$$dW - \nabla W \cdot d\vec{r} = 0 \implies (\vec{F} - \nabla W) \cdot d\vec{r} = 0$$

$$\implies \vec{F} = \nabla W$$

$$\cdots\cdots$$

Finding Potential
for Conservative Force Field

1. Choose a reference point A.
2. Choose a convenient path.
3. Integrate $\vec{F} \cdot d\vec{r}$ along this path to a variable point $B(x, y, z)$.

$$W = \int_A^B \vec{F} \cdot d\vec{r}$$

The integral is the value of W (including an additive constant) at point B which is an arbitrary point.

Gravitational field (choose $\hat{z}$ upward):

$$\vec{F} = -mg\hat{z}$$

Work done: $W = \int_0^z (-mg)\hat{z} \cdot dz'\hat{z} = -mgz$

Change in PE: $\Delta U = U(z) - U(0) = -W$

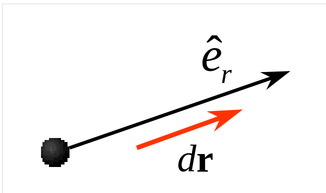
Force: $\vec{F} = \nabla W = -\nabla U$

Example

Potential due to a point charge q at the origin

$$\vec{E} = \frac{kq}{r^2}\hat{e}_r = \frac{kq}{r^3}\vec{r}$$

1. Choose $\phi = 0$ at $r \to \infty$
2. Choose a path of integral along the radial direction.
3. Use spherical coordinates



$$\vec{E} \cdot d\vec{r} = \frac{kq}{r^3}\vec{r} \cdot d\vec{r} = \frac{kq}{r^2}dr$$

$$\phi = -\int \vec{E} \cdot d\vec{r} = -kq\int_{\infty}^{r} \frac{dr}{r^2} = kq\left(\frac{1}{r}\right)_{\infty}^{r}$$

$$\phi(r) = \frac{kq}{r}$$