PC1134 Lecture 19

Topic

Double and triple integrals

Applications

- Area of surface
- Mass
- Centre of gravity
- Moment of inertia

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Scope

- Double integrals
- Calculation of double integral
- Triple integrals
- Calculation of Triple integral
- Polar coordinates
- Cylindrical coordinates
- Spherical coordinates

Double Integral

Consider $f(x,y)$,

$$\int f(x,y)dx \text{ is a function of } y.$$

Let

$$g(y) = \int f(x,y)dx$$

then

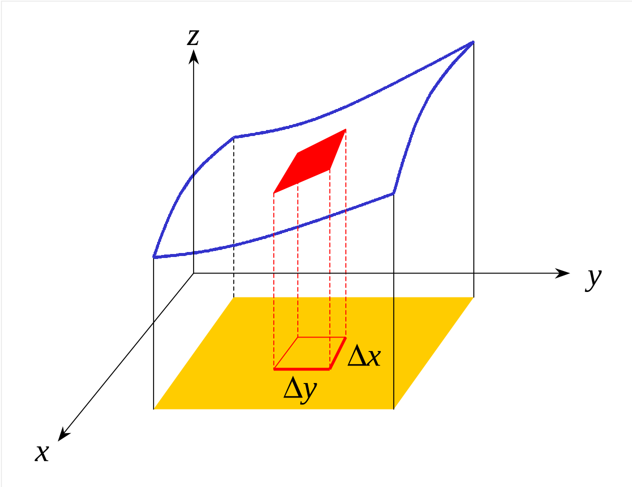
$$\int g(y)dy = \int dy \int dx f(x,y)$$

or

$$\int\int f(x,y)dxdy$$

Double Integral

$\int\int f(x,y)dxdy$ is the volume under the surface $f(x,y)$.



Divide volume into thin columns.

Base area of a column: $\Delta x \Delta y$

Volume of a column: $\Delta V = \Delta x \Delta y f(x,y)$

Total volume

$$V = \sum f(x_i, y_i) \Delta x_i \Delta y_i$$

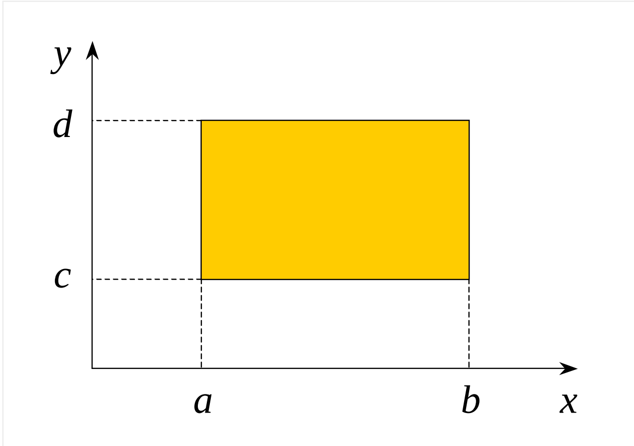
When $\Delta x_i \to 0$, $\Delta y_i \to 0$,

$$V = \int\int f(x,y)dxdy$$

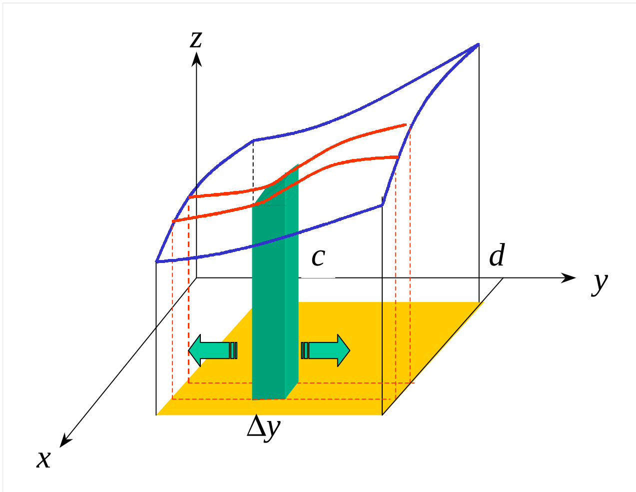
Calculating Double Integral

Case 1:

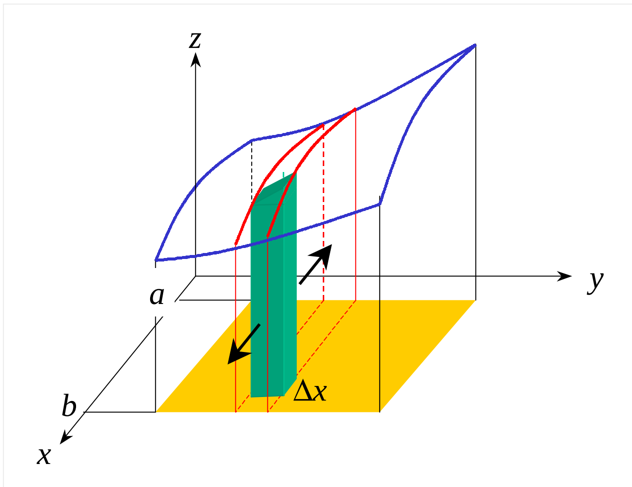
Rectangular domain



$$\int\int f(x,y)dxdy$$



$$= \int_a^b dx \int_c^d dy f(x,y)$$

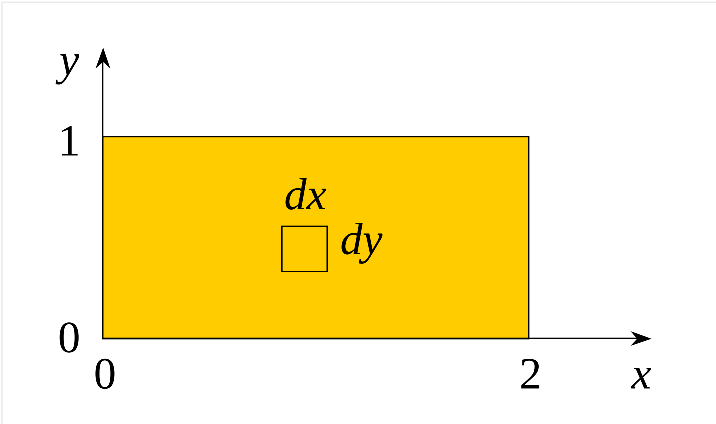


$$= \int_c^d dy \int_a^b dx f(x,y)$$

Example

Find the mass of a rectangular plate bounded by $x = 0$, $x = 2$, $y = 0$, $y = 1$ if its density is

$$f(x, y) = xy.$$



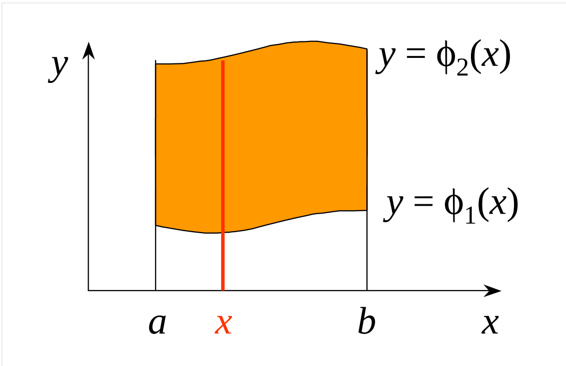
$$dA = dxdy$$

$$dM = f(x, y)dxdy$$

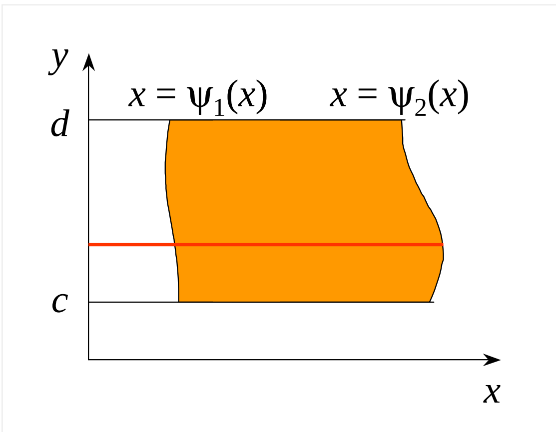
$$\begin{aligned} M &= \int dM = \int\int xydxdy \\ &= \int_{x=0}^{2}\int_{y=0}^{1} xydxdy = \int_0^2 xdx \int_0^1 ydy = 1 \end{aligned}$$

Calculating Double Integral

Case 2: $f(x, y)$ is defined over an arbitrary region.



$$\int\int f(x, y)dxdy = \int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy$$



$$\int\int f(x, y)dxdy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx$$

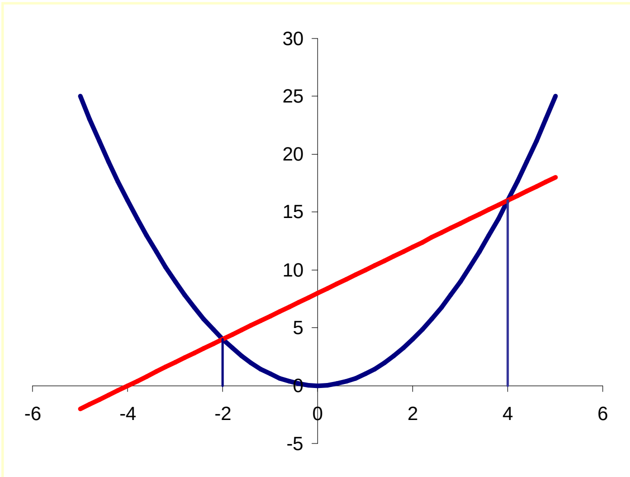
Example

Evaluate $\int\int_A x dx dy$ where A is the area between the parabola $y = x^2$ and the straight line $2x - y + 8 = 0$.

$$y_1 = x^2$$
$$y_2 = 2x + 8$$

Intersection

$$y_1 = y_2$$
$$x^2 = 2x + 8$$
$$\Downarrow$$
$$x^2 - 2x - 8 = 0$$
$$x_a = -2$$
$$x_b = 4$$

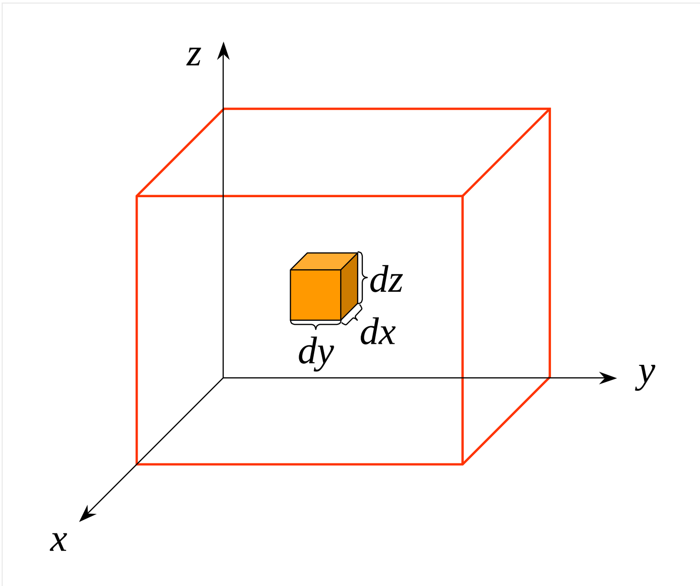


$$\begin{aligned}
\int\int_A x dx dy &= \int_{-2}^{4} dx x \int_{x^2}^{2x+8} dy \\
&= \int_{-2}^{4} dx x \left[(2x+8) - x^2\right] \\
&= \int_{-2}^{4} \left[-x^3 + 2x^2 + 8x\right] dx \\
&= \left(-\frac{x^4}{4} + \frac{2}{3}x^3 + 4x^2\right)_{-2}^{4} = 36
\end{aligned}$$

Triple Integral

$$\int\int\int f(x,y,z)\,\underbrace{dx\,dy\,dz}_{dV}$$

(volume element)



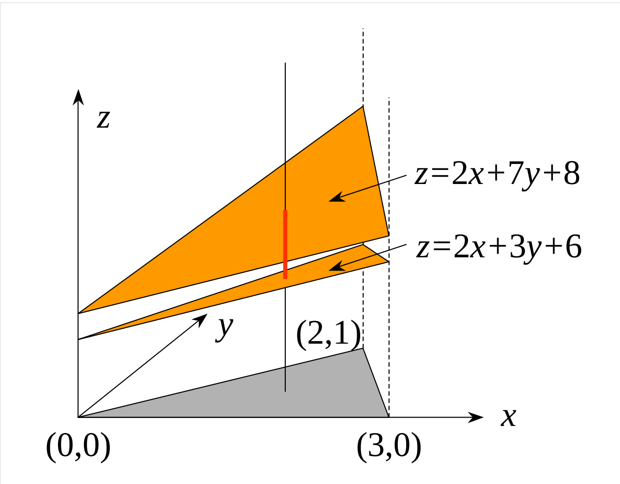
Calculation:

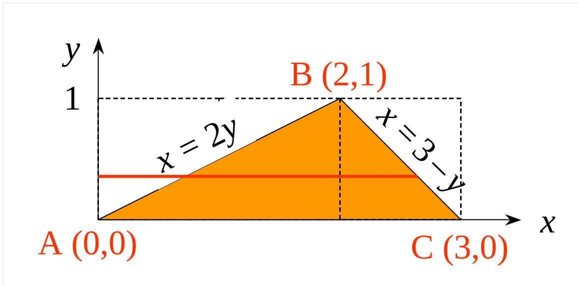
- All limits are constants:
  integration can be done in any order.

- Complicated volume:
  geometry $\rightarrow$ order of integration.

Example

Find the volume between the planes $z = 2x + 3y + 6$ and $z = 2x + 7y + 8$ and over the triangle with vertices at $(0, 0)$, $(3, 0)$ and $(2, 1)$.





$$V = \int\int\int dx\,dy\,dz = \int_0^1 dy \int_{2y}^{3-y} dx \int_{2x+3y+6}^{2x+7y+8} dz$$

$$= \int_0^1 dy \int_{2y}^{3-y} dx(2x + 7y + 8 - 2x - 3y - 6)$$

$$= 2\int_0^1 dy \int_{2y}^{3-y} (2y+1)dx = 2\int_0^1 dy(2y+1)(3-y-2y)$$

$$= 6\int_0^1 dy(2y + 1)(1 - y) = 6\int_0^1 (-2y^2 + y + 1)dy = 5$$