PC1134 Lecture 20

Topic:

Calculations of multiple integrals in cylindrical and spherical coordinate systems.

Relevance:

Many physical systems have cylindrical or spherical symmetry. Double and triple integrations can be done more conveniently using cylindrical or spherical corodinates.

Applications

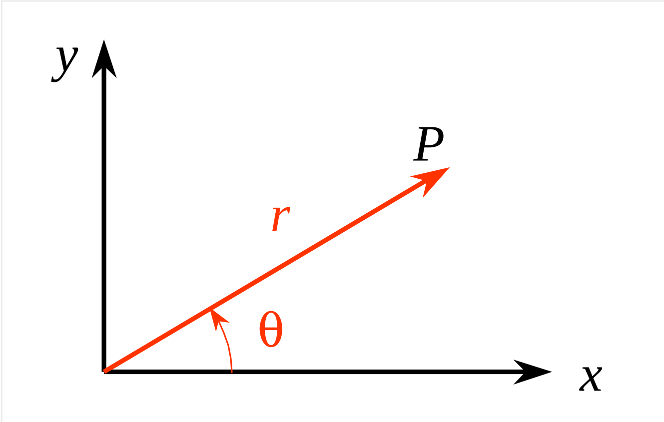
- Area
- Volume
- Mass
- Centre of mass
- Moment of inertia

Scope

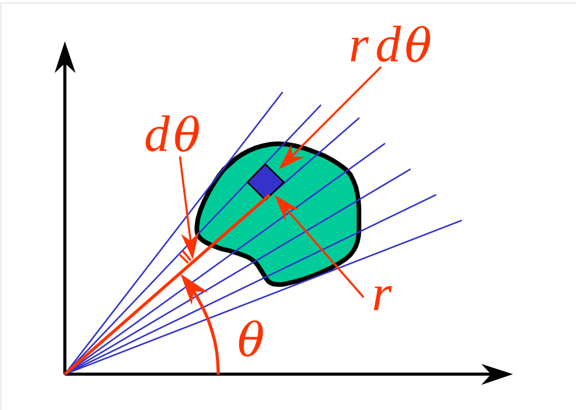
- Polar coordinates
- Cylindrical coordinates
- Spherical coordinates

Polar Coordinates

$$(x, y)$$
$$(r, \theta)$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$



$$f(x, y) = f(r\cos\theta, r\sin\theta) = f(r, \theta)$$



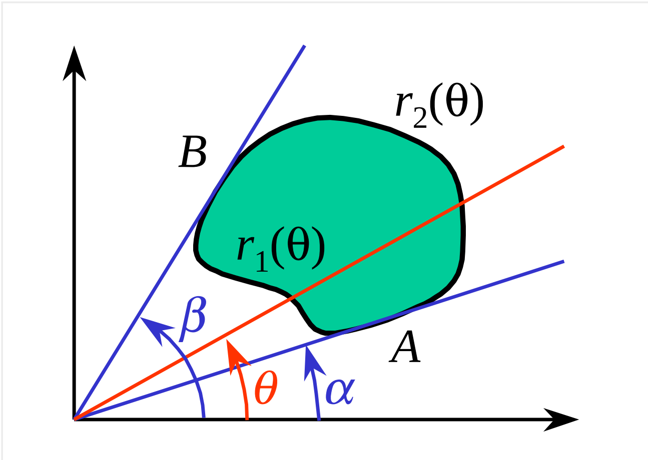
Area element:

$$dA = dr \cdot r d\theta = r dr d\theta$$

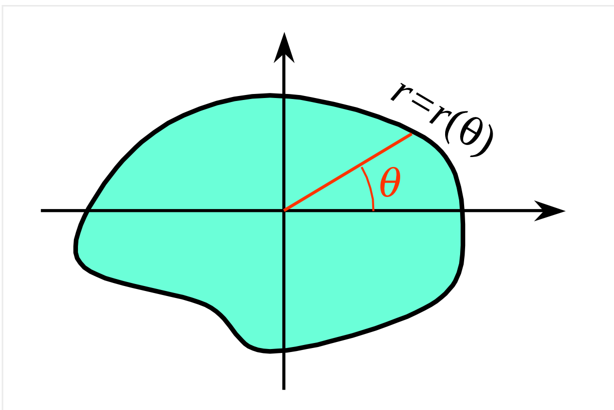
$$\int\int_A f(x, y) dx dy$$
$$\int\int_A f(r, \theta) r dr d\theta$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dxdy \Longrightarrow r dr d\theta$$

Polar Coordinates



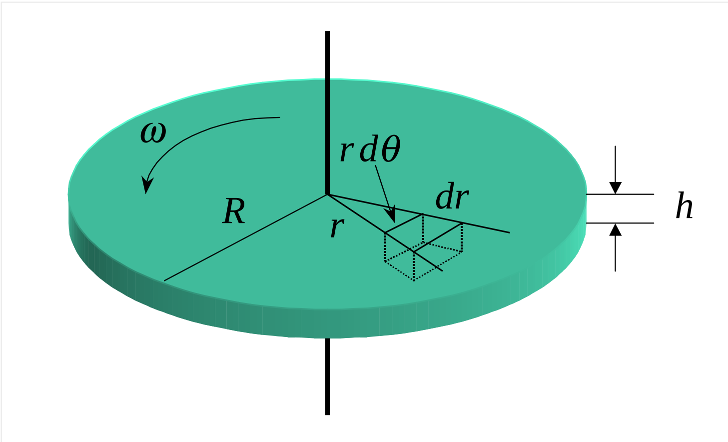
$$\int\int f(r,\theta) r dr d\theta = \int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r dr d\theta$$



$$\int\int f(r,\theta) r dr d\theta = \int_{0}^{2\pi}\int_{0}^{r(\theta)} f(r,\theta) r dr d\theta$$

Example

Moment of inertia of a circular disk (density ρ)



Consider a small element

Volume: $dV = dr(rd\theta)h$

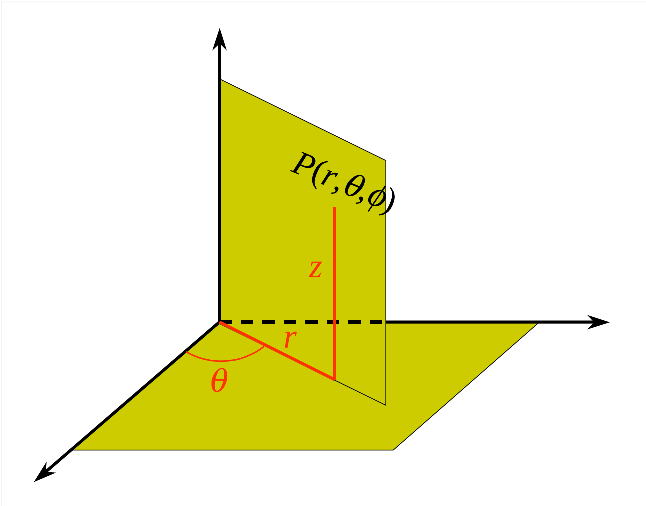
Mass: $dm = \rho dV = \rho hrdrd\theta$

Moment of inertia: $dmr^2 = \rho hr^3drd\theta$

$$\begin{aligned} I &= \int dmr^2 = \rho h \int_0^R r^3 dr \int_0^{2\pi} d\theta \\ &= \rho h \frac{R^4}{4} 2\pi = \frac{1}{2} MR^2 \end{aligned}$$

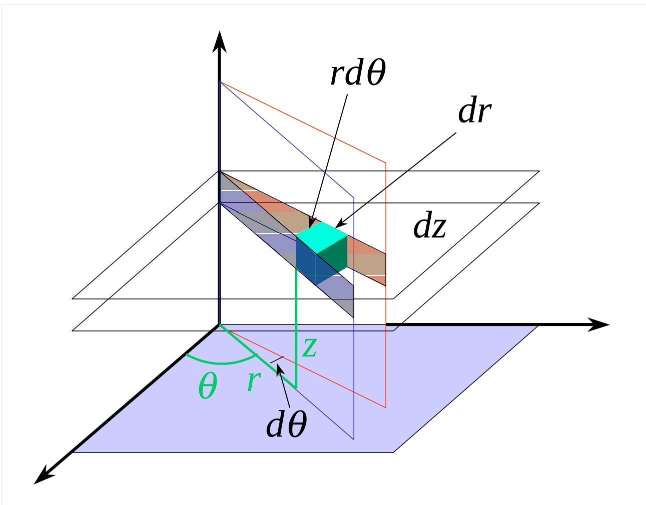
$$(M = \pi R^2 h\rho)$$

Cylindrical Coordinates



$$(x, y, z) \longrightarrow (r, \theta, z)$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

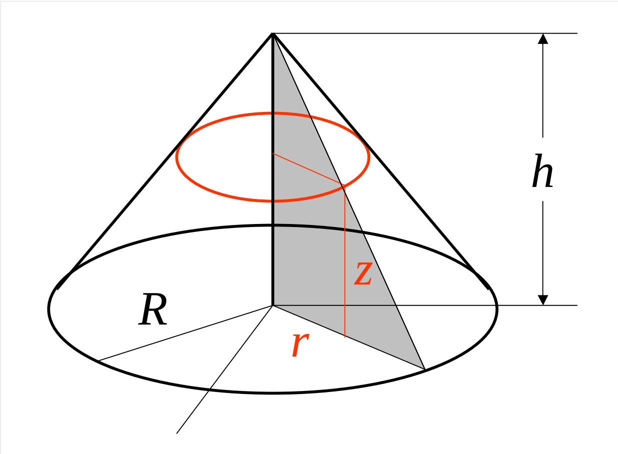


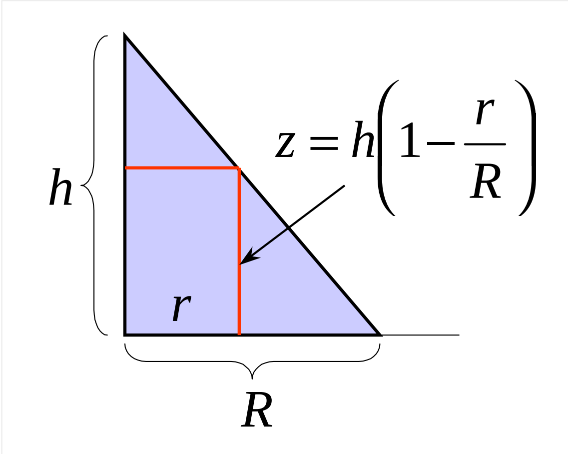
$$\begin{aligned} dv &= dxdydz \\ &\Rightarrow dr(rd\theta)dz \\ &= rdrd\theta dz \end{aligned}$$

$$\iiint f(x, y, z)dv \Longrightarrow \iiint f(r, \theta, z)rdrd\theta dz$$

Example

Volume of a solid cone:

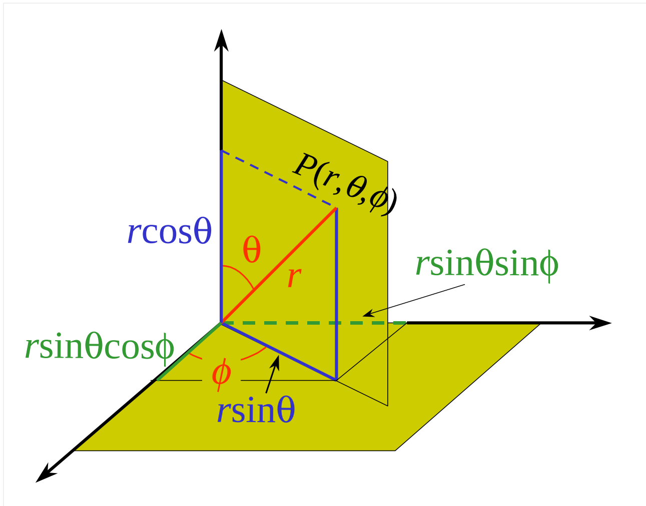




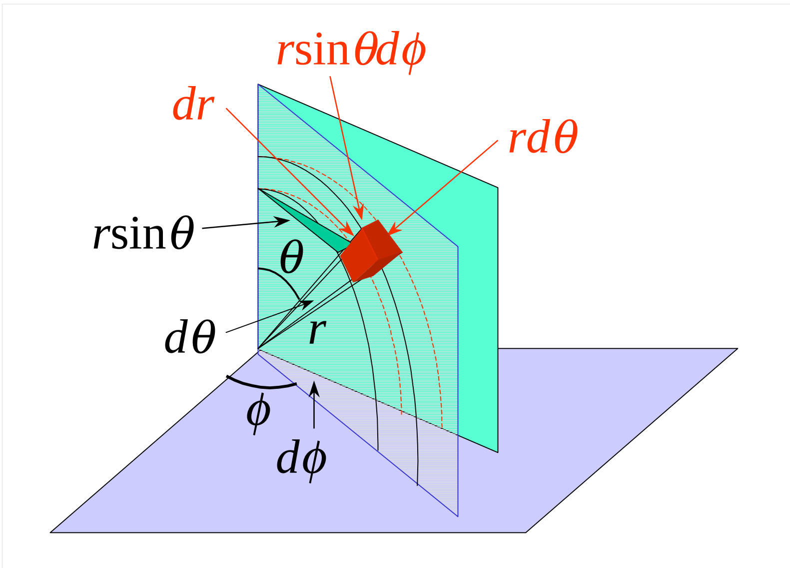
$$
\begin{aligned}
V &= \iiint dv = \iiint r dr d\theta dz \\
&= \int_0^{2\pi} d\theta \int_0^R r dr \int_0^{h(1-r/R)} dz \\
&= 2\pi \int_0^R r dr h(1 - r/R) \\
&= 2\pi h \int_0^R h(r - r^2/R) dr \\
&= 2\pi h[R^2/2 - R^3/(3R)] = \pi R^2 h/3
\end{aligned}
$$

Spherical Coordinates

$$(x, y, z) \Longleftrightarrow (r, \theta, \phi)$$



$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

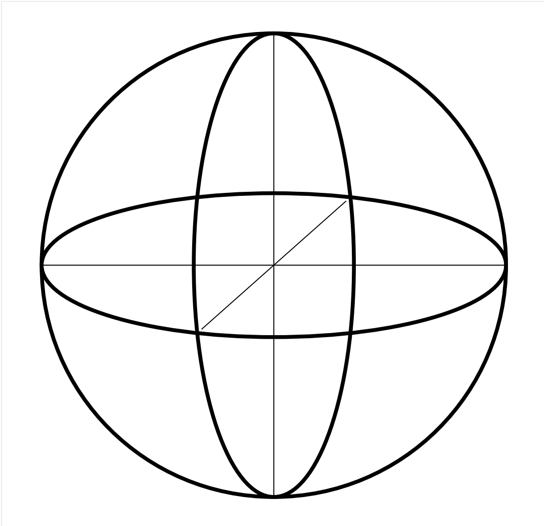


$$dV = dr(rd\theta)(r\sin\theta d\phi) = r^2\sin\theta dr d\theta d\phi$$

$$\iiint f(x, y, z)dxdydz \Longrightarrow \iiint f(r, \theta, \phi)r^2\sin\theta dr d\theta d\phi$$

Example

Moment of inertia of a uniform sphere about its axis



$$x^2 + y^2 + z^2 = R^2$$

$$I_x = \iiint (y^2 + z^2)\rho dv$$

$$I_y = \iiint (z^2 + x^2)\rho dv$$

$$I_z = \iiint (x^2 + y^2)\rho dv$$

By symmetry, $I_x = I_y = I_z = I$.

$$\begin{aligned}
3I &= 2\rho \iiint (x^2 + y^2 + z^2) dv \\
&= 2\rho \iiint r^2 dv = 2\rho \iiint r^4 \sin\theta dr d\theta d\phi \\
&= 2\rho \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^R r^4 dr \\
&= 2\rho 2\pi \left(-\cos\theta|_0^{\pi}\right) \frac{1}{5} R^5 = \frac{8}{5}\pi\rho R^5 = \frac{2}{5} M R^2
\end{aligned}$$

General Coordinate Transformation

Polar	$dA = dxdy$	$\Longrightarrow$	$rdrd\theta$
Cylindrical	$dV = dxdydz$	$\Longrightarrow$	$rdrd\theta dz$
Spherical	$dV = dxdydz$	$\Longrightarrow$	$r^2 \sin\theta drd\theta d\phi$

2-dimensional coordinate transformation

$$x = x(u, v)$$

$$y = y(u, v)$$

$$dxdy \Longrightarrow |J|dudv$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

General Coordinate Transformation

3-dimensional coordinate transformation

$$x = x(u, v, w)$$
$$y = y(u, v, w)$$
$$z = z(u, v, w)$$

$$dxdydz = |J|dudvdw$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$