PC1134 Lecture 21

Topic:

Surface Integrals

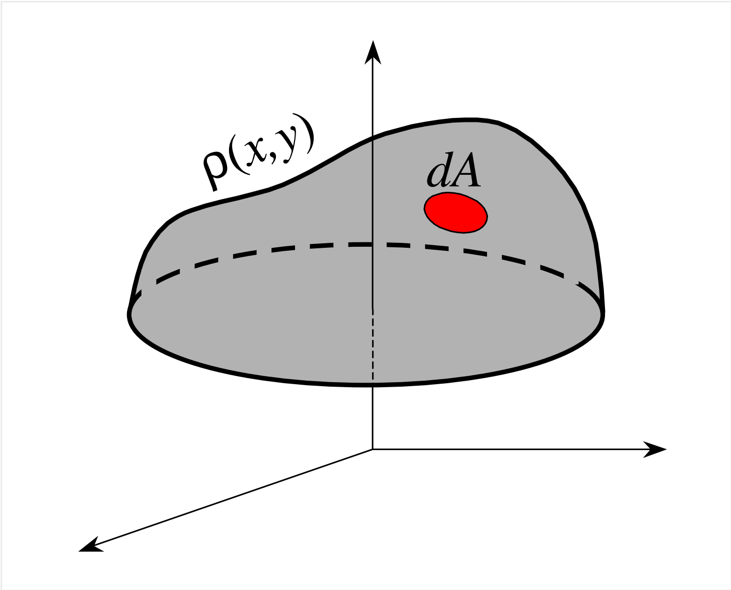
Applications:

Surface area
Moment of surface
Other integration over surfaces

Scope

- Surface integrals
- Evaluation of surface integrals

Surface Integral



Given area density $\rho(x, y)$ (mass per unit area),

$$M = ?$$

Divide the surface into small regions, if ΔA_i is the area of the ith region, then

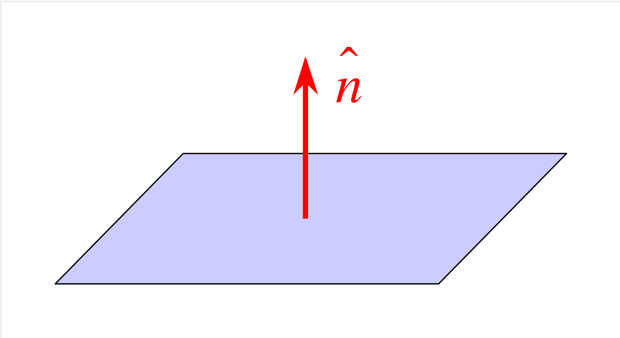
$$M \approx \sum_i \rho(x, y) \Delta A_i$$

When $\Delta A_i \to 0$,

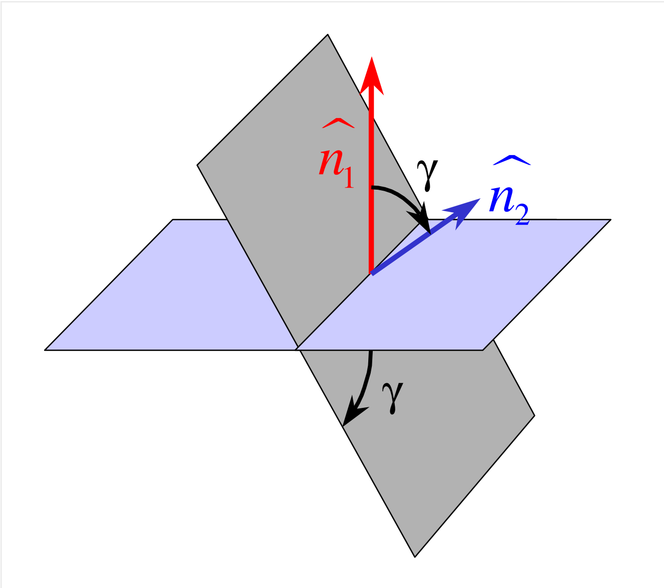
$$M = \iint \rho(x, y) dA$$

Orientation of a Surface (Plane)

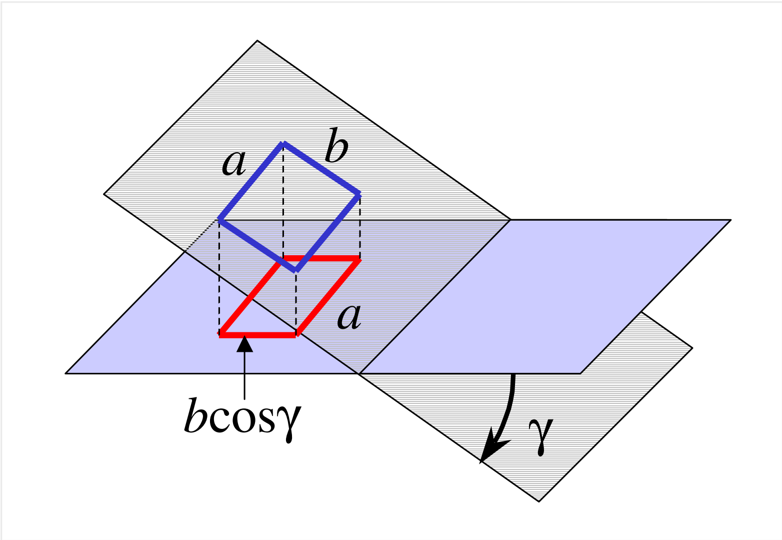
Orientation of a surface can be specified by the normal — a unit vector perpendicular to the surface (plane).



If two planes make an angle of γ, then the angle between their normals is γ.



Projection



Area of rectangle in inclined plane:

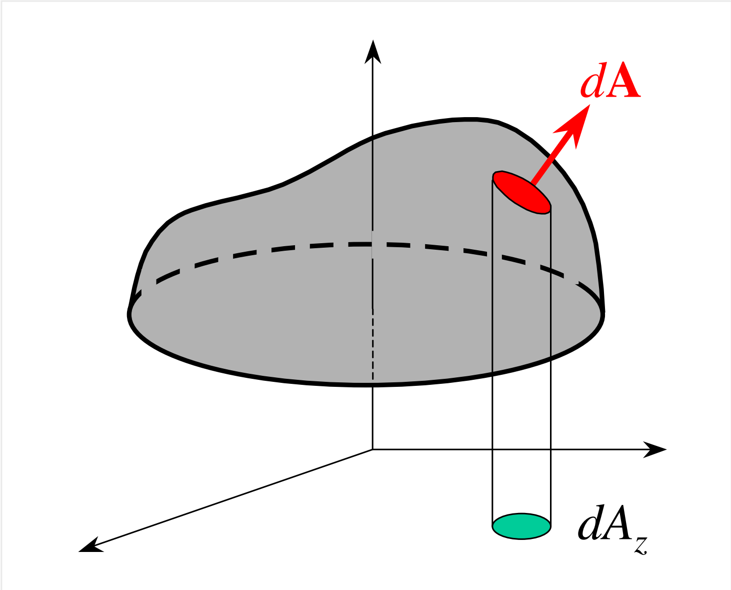
$$A = ab$$

Area of projection in horizontal plane:

$$A_z = ab\cos\gamma = A\cos\gamma$$

The above also holds for any shaped area.

Surface Element



The vector differential $d\vec{A}$ can be used to represent a vector area element of a surface.

$$d\vec{A} = \hat{n} dA, \quad dA_z = d\vec{A} \cdot \hat{z}$$

$\hat{n}$ is a unit normal to the surface at the position of the element and dA is the (scalar) area of the element.

For a closed surface, the direction of $\hat{n}$ is taken to point outwards from the enclosed volume. For an open surface, the direction of $\hat{n}$ is given by the right-hand sense with respect to the direction in which the perimeter is traversed.

Evaluation of Surface Integral

$$I = \iint f(x,y)dA = \iint f(x,y)\frac{dA_z}{\cos\gamma} = \iint f(x,y)\frac{dxdy}{\cos\gamma}$$

A. Equation of the surface is $\phi(x,y,z) = \text{const}$

$\nabla\phi$ points to its normal direction but $\nabla\phi \neq 1$.

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\cos\gamma = \hat{n}\cdot\hat{k} = \frac{\hat{k}\cdot\nabla\phi}{|\nabla\phi|} = \frac{\partial\phi/\partial z}{|\nabla\phi|}$$

$$|\nabla\phi| = \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]^{1/2}$$

$$\cos\gamma = \frac{\partial\phi/\partial z}{\sqrt{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2}}$$

$$I = \iint f(x,y)\frac{\sqrt{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2}}{\partial\phi/\partial z}dxdy$$

Evaluation of Surface Integral

$$I = \iint F(x, y) dA = \iint F(x, y) \frac{dx dy}{\cos \gamma}$$

B. Equation of the surface is $z = f(x, y)$

This can be written as

$$\phi(x, y, z) = z - f(x, y) = 0$$

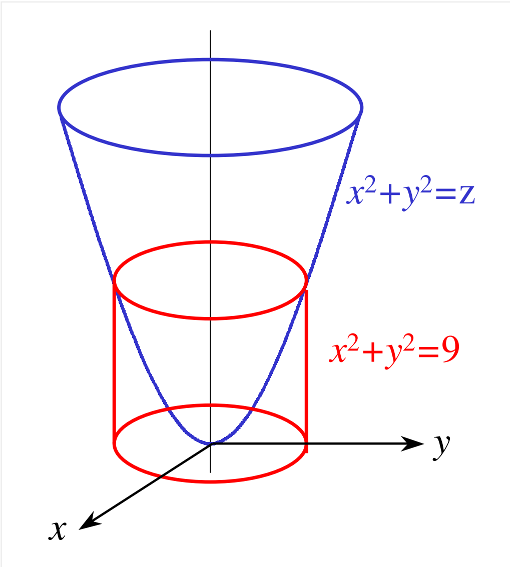
$$\frac{\partial \phi}{\partial z} = 1, \quad \frac{\partial \phi}{\partial x} = -\frac{\partial f}{\partial x}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial f}{\partial y}.$$

$$\cos \gamma = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

$$I = \iint F(x, y) \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dx dy$$

Example

Area of the paraboloid $x^2 + y^2 = z$ inside the cylinder $x^2 + y^2 = 9$.



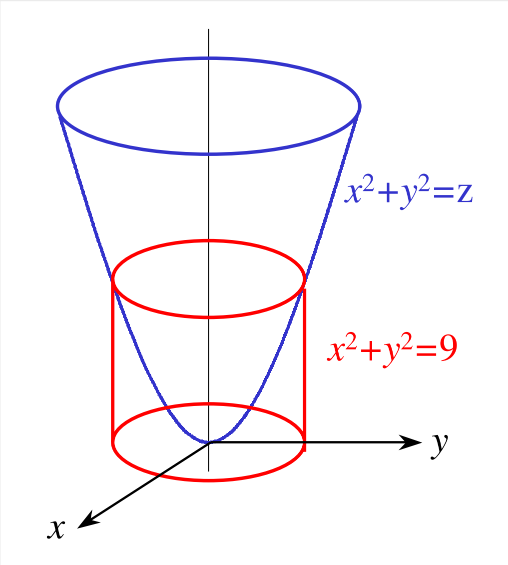
$$\phi(x, y, z) = z - x^2 - y^2 = 0$$

$$\frac{\partial \phi}{\partial x} = -2x, \quad \frac{\partial \phi}{\partial y} = -2y, \quad \frac{\partial \phi}{\partial z} = 1.$$

$$A = \iint dA = \iint \sqrt{4x^2 + 4y^2 + 1}\, dx dy$$

Example (cont.)

$$A = \iint dA = \iint \sqrt{4x^2 + 4y^2 + 1}\; dxdy$$



$$\begin{aligned} (x, y) &\rightarrow (r, \theta) \\ dxdy &\rightarrow r dr d\theta \end{aligned}$$

$$\begin{aligned} A &= \iint \sqrt{4r^2 + 1}\; r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^3 \sqrt{4r^2 + 1}\; r dr \\ &= \frac{\pi}{6} \left[(37)^{3/2} - 1 \right] \end{aligned}$$