PC1134 Lecture 22

Topic:

Green's theorem in the plane

Applications:

Relation between a double integral and line integral along the boundary of the area.

Evaluate either a line integral around a closed path or a double integral over the area enclosed, whichever is easier to do.

Scope

- Green's Theorem in a plane
- Proof
- Examples

Green's Theorem

If

1. the region of integration is *simple*; and
2. $P = P(x, y)$ and $Q = Q(x, y)$ have continuous first partial derivatives at every point of this region.

then

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint Pdx + Qdy$$

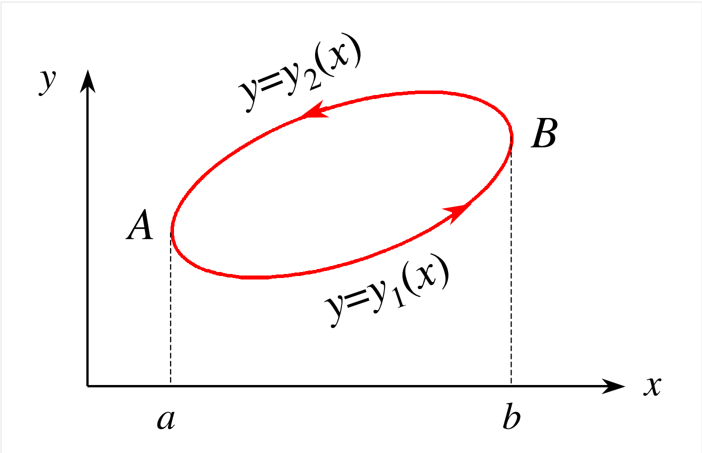
$\oint$ means integration around a closed curve back to the starting point.

The line integral is in the counter-clockwise direction.

Proof

$$\iint \frac{\partial P}{\partial y} dx dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} dy$$

$$= \int_a^b [P(x, y_2) - P(x, y_1)]\, dx$$



$$\oint P dx = \int_{AB} P dx + \int_{BA} P dx$$

$$= \int_a^b P[x, y_1(x)] dx + \int_b^a P[x, y_2(x)] dx$$

$$= -\int_a^b P[x, y_2(x)] dx - \int_b^a P[x, y_1(x)] dx$$

$$\iint \frac{\partial P}{\partial y} dx dy = -\oint P dx$$

Proof (cont.)

$$\iint \frac{\partial P}{\partial y} dx dy = - \oint P dx$$

Similarly it can be shown that

$$\iint \frac{\partial Q}{\partial x} dx dy = \oint Q dy$$

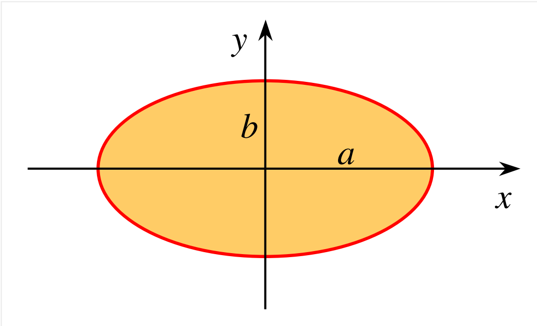
$$\Downarrow$$

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint P dx + Q dy$$

Example

Area of an ellipse

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}$$



$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint Pdx + Qdy$$

Choose $Q = x$, $P = -y$,

$$\begin{aligned} \iint dxdy &= \frac{1}{2}\oint xdy - ydx \\ &= \frac{1}{2}\oint a\cos\theta b\cos\theta d\theta + b\sin\theta a\cos\theta d\theta \\ &= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)d\theta \\ &= \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab \end{aligned}$$

Example

Let

$$P = F_x, \quad Q = F_y$$

$$\oint P dx + Q dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\oint (F_x dx + F_y dy) = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy$$

$$\oint \vec{F} \cdot d\vec{r} = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy$$

If

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

then

$$\oint \vec{F} \cdot d\vec{r} = 0 \quad \Longrightarrow \quad \text{conservative force}$$

$\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}$ is the z-component of $\nabla \times \vec{F}$. In 2D,

$$\nabla \times \vec{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$