PC1134 Lecture 23

Topic:

Divergence theorem and applications

Relevance:

- The divergence theorem converts a volume (triple) integral into a integral over a closed surface and vice versa. We can then evaluate whichever one is easier to do.
- Very important applications in electricity (Gauss' law), fluid dynamics (equation of continuity), heat flow, flow of particles, etc.

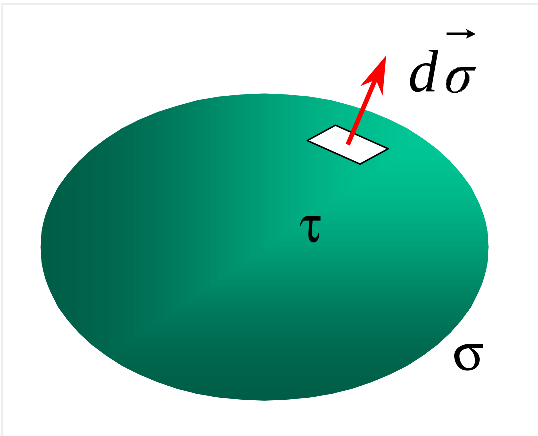
Scope

- The theorem
- Proof
- Equation of continuity
- Gauss's law
- Examples

Divergence Theorem

The normal surface integral of a function $\vec{V}$ over the boundary of a closed surface of arbitrary shape is equal to the volume (triple) integral of the divergence of $\vec{V}$ taken throughout the enclosed volume.

$$\iint_\sigma \vec{V} \cdot d\vec{\sigma} = \iiint_\tau \nabla \cdot \vec{V} d\tau$$



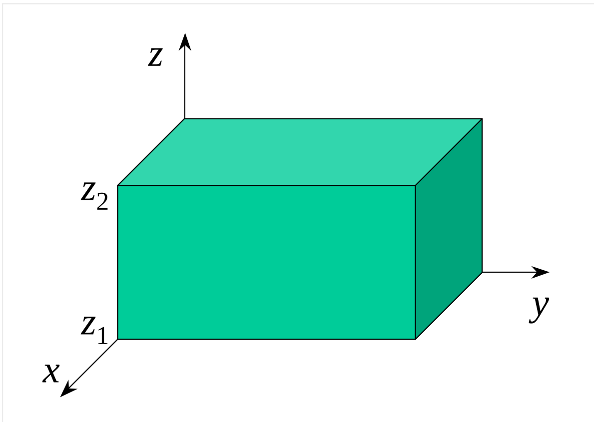
$\iint$: integral over closed surface
$d\tau$: volume element
$d\vec{\sigma}$: surface element
Magnitude = area
Direction: normal

Proof of Divergence Theorem

$$\iint_\sigma \vec{F} \cdot d\vec{\sigma} = \iiint_\tau \nabla \cdot \vec{F} d\tau$$

$$RHS = \iiint \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dxdydz$$

Consider the volume shown below

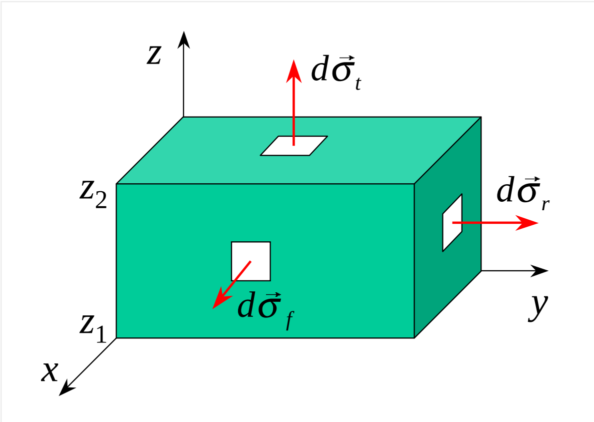


Integrate the last term with respect to z from z_1 to z_2

$$\iiint \frac{\partial F_z}{\partial z} dxdydz = \iint_\sigma \left[F_z(x, y, z_2) - F_z(x, y, z_1) \right] dxdy$$

$$= \iint_{\sigma_t} F_z(x, y, z_2) dxdy - \iint_{\sigma_b} F_z(x, y, z_1) dxdy$$

Proof of Divergence Theorem (cont.)



Bottom	$d\vec{\sigma}_b = -dxdy\hat{k}$
Front	$d\vec{\sigma}_f = dydz\hat{i}$
Left	$d\vec{\sigma}_l = -dzdx\hat{j}$
Top	$d\vec{\sigma}_t = dxdy\hat{k}$
Back	$d\vec{\sigma}_{bk} = -dydz\hat{i}$
Right	$d\vec{\sigma}_r = dzdx\hat{j}$

$$\hat{k} \cdot d\vec{\sigma}_t = dxdy$$

$$\hat{k} \cdot d\vec{\sigma}_b = -dxdy$$

$$\iiint \frac{\partial F_z}{\partial z} dxdydz$$

$$= \iint_{\sigma_t} F_z(x, y, z_2)\hat{k} \cdot d\vec{\sigma}_t + \iint_{\sigma_b} F_z(x, y, z_1)\hat{k} \cdot d\vec{\sigma}_b$$

For all other surfaces (sides): $\hat{k} \cdot d\vec{\sigma} = 0$

$$\iiint \frac{\partial F_z}{\partial z} dxdydz = \iint_{\sigma} F_z(x, y, z)\hat{k} \cdot d\vec{\sigma}$$

Proof of Divergence Theorem (cont.)

Similar integrations can be done to get

$$\iiint \frac{\partial F_x}{\partial x} dxdydz = \iint\limits_{\sigma} F_x(x,y,z)\hat{i} \cdot d\vec{\sigma}$$

$$\iiint \frac{\partial F_y}{\partial y} dxdydz = \iint\limits_{\sigma} F_y(x,y,z)\hat{j} \cdot d\vec{\sigma}$$

$$\iiint \frac{\partial F_z}{\partial z} dxdydz = \iint\limits_{\sigma} F_z(x,y,z)\hat{k} \cdot d\vec{\sigma}$$

Combine the three equations

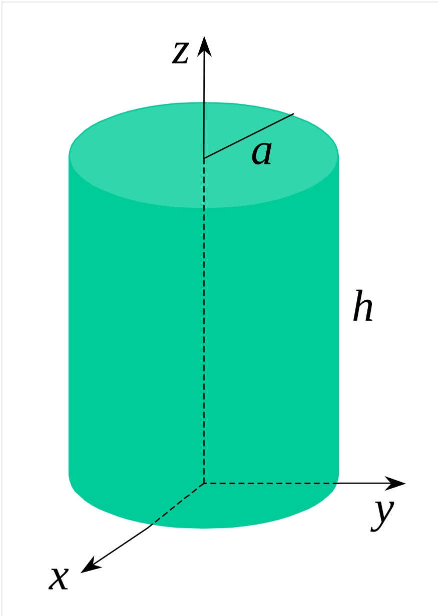
$$\iiint \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dxdydz = \iint\limits_{\sigma} \left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \right) \cdot d\vec{\sigma}$$

Therefore

$$\iiint_{\tau} \nabla \cdot \vec{F} d\tau = \iint\limits_{\sigma} \vec{F} \cdot d\vec{\sigma}$$

Example

Let $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$, evaluate $\iint \vec{V} \cdot d\vec{\sigma}$ over the closed surface of the cylinder shown below.



$$\iint_{\sigma} \vec{V} \cdot d\vec{\sigma} = \iiint_{\tau} \nabla \cdot \vec{V} d\tau$$

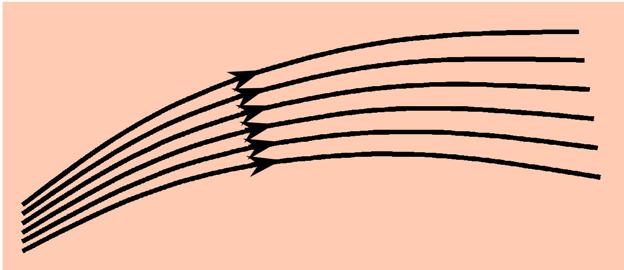
$$\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \cdot \vec{V} = 1 + 1 + 1 = 3$$

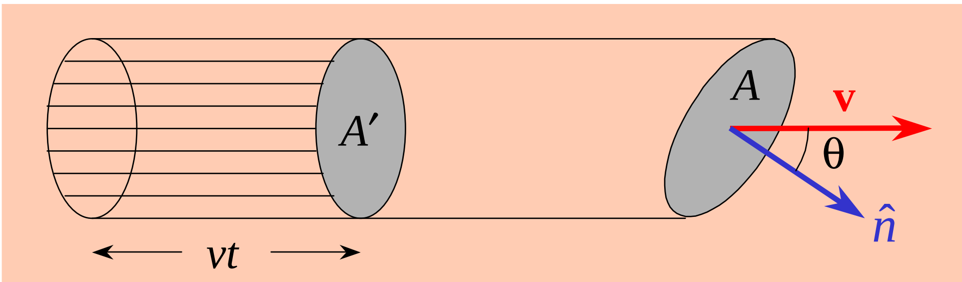
$$\iint_{\sigma} \vec{V} \cdot d\vec{\sigma} = \iiint_{\tau} \nabla \cdot \vec{V} d\tau = 3 \iiint_{\tau} d\tau = 3\pi a^2 h$$

Flow of Substance

Water, gas, heat, radioactive particles, electric and magnetic fields, etc.

Stream line: 

Assume density of fluid is ρ, velocity of fluid is $\vec{v}$.



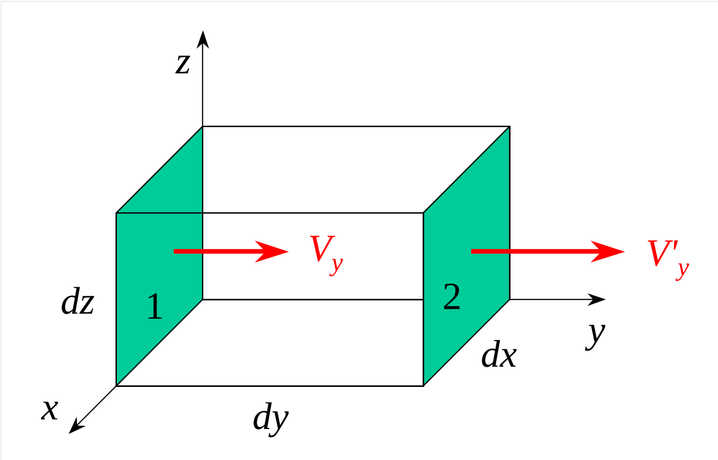
Amount of fluid crossing an area A' in time t is

$$(vt)A'\rho \qquad \vec{v} \text{ is } \perp \text{ cross section}$$
$$(vt)A\rho\cos\theta \qquad \vec{v}\cdot\hat{n} = v\cos\theta$$

Amount of fluid cross unit area of the surface in unit time is

$$v\rho\cos\theta = V\cos\theta = \vec{V}\cdot\hat{n} \quad (V = \rho v)$$

Physical Meaning of Divergence



Assume that fluid flows from left to right and enters the left surface with velocity V_y and leaves the right surface with velocity V'_y. Amount of fluid flow into the volume through surface 1 is

$$(\vec{V} \cdot \hat{j}) dxdz = V_y dxdz$$

Amount of fluid flow out of the volume through surface 2 is

$$V'_y dxdz = \left(V_y + \frac{\partial V_y}{\partial y} dy \right) dxdz$$

Net increase in mass

$$V_y dxdz - \left(V_y + \frac{\partial V_y}{\partial y} dy \right) dxdz = -\frac{\partial V_y}{\partial y} dxdydz$$

Physical Meaning of Divergence (cont.)

Similarly, net mass increase due to flow in the x-direction and the z-direction are

$$-\frac{\partial A_x}{\partial x}dxdydz, \ \& \ -\frac{\partial A_z}{\partial z}dxdydz.$$

respectively. Total increase in mass per unit volume per unit time is

$$\frac{1}{dxdydz}\left(-\frac{\partial A_x}{\partial x}-\frac{\partial A_y}{\partial y}-\frac{\partial A_z}{\partial z}\right)dxdydz$$

$$= -\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right) = -\nabla\cdot\vec{A}$$

If there is no source in the volume, this must be equal to the increase of the fluid mass per unit time per unit volume — rate of density increase

$$-\nabla\cdot\vec{A} = \frac{\partial\rho}{\partial t}$$

$$\nabla\cdot\vec{A} + \frac{\partial\rho}{\partial t} = 0$$

(Equation of continuity)

Gauss' Law

Electric field: $\vec{E}$

Maxwell's equation: $\nabla \cdot \vec{E} = \rho/\varepsilon_0$

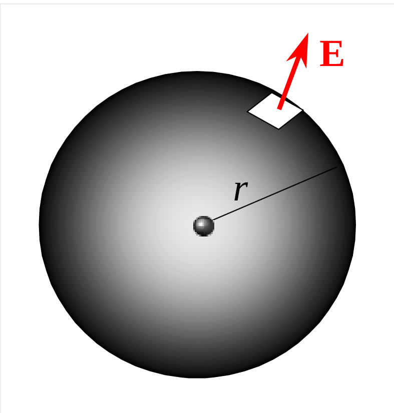
ρ: charge density
ε_0: constant

$$\iint\limits_{\sigma} \vec{E} \cdot d\vec{\sigma} = \iiint\limits_{\tau} \nabla \cdot \vec{E} d\tau = \frac{1}{\varepsilon_0} \iiint\limits_{\tau} \rho d\tau$$

$$\iint\limits_{\sigma} \vec{E} \cdot d\vec{\sigma} = \frac{Q}{\varepsilon_0}$$

$Q = \iiint\limits_{\tau} \rho d\tau$ is the total charge inside σ.

Point Charge



$$\iint\limits_{\sigma} \vec{E} \cdot d\vec{\sigma} = \iint\limits_{\sigma} E d\sigma$$

$$= E \iint\limits_{\sigma} d\sigma = 4\pi r^2 E = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2}$$