PC1134 Lecture 24

Topic:

Stokes theorem and applications

Relevance:

- The Stokes theorem converts a surface integral to a line integral and vice versa. We can then evaluate whichever one that is easier.
- Stokes theorem has very important applications in magnetism and other physical problems.

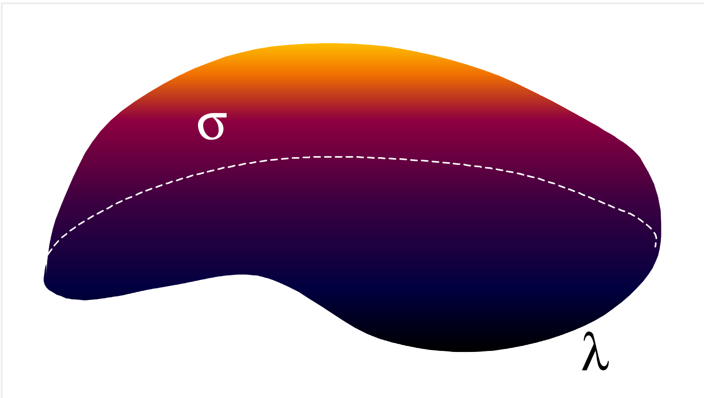
Scope

- The theorem
- Physical meaning of curl
- Application in magnetism (Ampére's law)
- Examples

Stokes Theorem

If $\vec{F}$ and its first derivatives are continuous, the line integral of $\vec{F}$ around a *closed curve* λ is equal to the normal surface integral of curl $\vec{F}$ over an *open surface bounded by* λ.

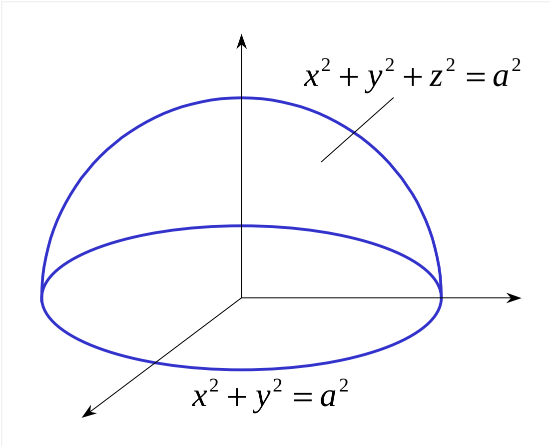
$$\oint_\lambda \vec{F} \cdot d\vec{\lambda} = \iint_\sigma \nabla \times \vec{F} \cdot d\vec{\sigma}$$



In other words, the surface integral of curl $\vec{F}$ taken over any open surface σ is equal to the line integral of $\vec{F}$ around the periphery λ of the surface.

Example

Given $\vec{V} = 4y\hat{i} + x\hat{j} + 2z\hat{k}$, find $\iint \nabla \times \vec{V} \cdot d\vec{\sigma}$ over the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.



Method 1: Evaluate the surface integral directly.

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 4y & x & 2z \end{vmatrix} = (1-4)\hat{k} = -3\hat{k}$$

$$d\vec{\sigma} = d\sigma \hat{e}_r$$

$$\nabla \times \vec{V} \cdot d\vec{\sigma} = (-3\hat{k}) \cdot \hat{e}_r d\sigma = -3\cos\theta d\sigma$$

$$d\sigma = r^2 \sin\theta d\theta d\phi$$

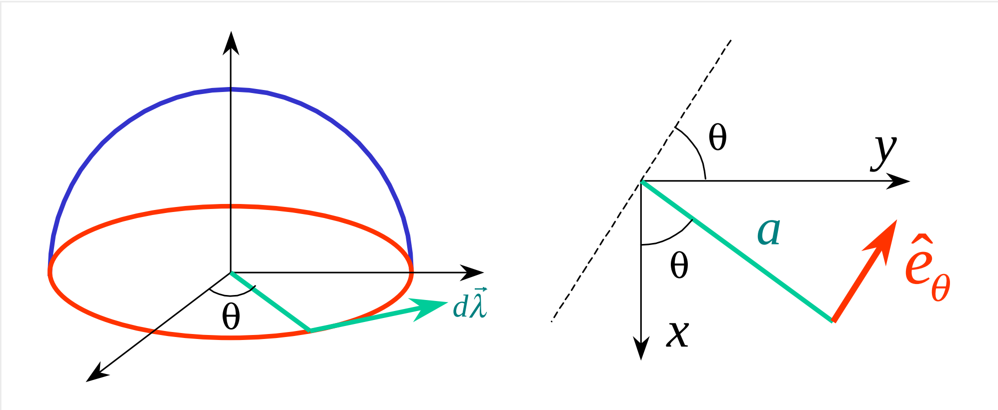
$$\iint \nabla \times \vec{V} \cdot d\vec{\sigma} = -3a^2 \iint \sin\theta\cos\theta d\theta d\phi$$

$$= -3a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\cos\theta d\theta = -3a^2 \cdot 2\pi \cdot \frac{1}{2} = -3\pi a^2$$

Example (cont.)

Method 2: Using Stokes theorem

$$\iint_\sigma \nabla \times \vec{V} \cdot d\vec{\sigma} = \oint_\lambda \vec{V} \cdot d\vec{\lambda}$$



$$\vec{V} = 4y\hat{i} + x\hat{j} + 2z\hat{k}, \quad d\vec{\lambda} = d\theta a\hat{e}_\theta$$

$$\begin{aligned}
\vec{V} \cdot d\vec{\lambda} &= ad\theta(4y\hat{i} \cdot \hat{e}_\theta + x\hat{j} \cdot \hat{e}_\theta + 2z\hat{k} \cdot \hat{e}_\theta) \\
&= a(-4a\sin\theta\sin\theta + a\cos\theta\cos\theta)d\theta \\
&= a^2(-4\sin^2\theta + \cos^2\theta) = a^2(1 - 5\sin^2\theta)d\theta
\end{aligned}$$

$$\oint_\lambda \vec{V} \cdot d\vec{\lambda} = \int_0^{2\pi} a^2(1 - 5\sin^2\theta)d\theta$$

$$= a^2\left(\int_0^{2\pi} d\theta - 5\int_0^{2\pi} \sin^2\theta d\theta\right) = -3\pi a^2$$

Example (cont.)

Method 3:

$$\iint\limits_{\text{hemisphere}} \nabla \times \vec{V} \cdot d\vec{\sigma} = \oint\limits_{\text{circle}} \vec{V} \cdot d\vec{\lambda} = \iint\limits_{\substack{\text{any surface}\\ \text{bounded by}\\ \text{the circle}}} \nabla \times \vec{V} \cdot d\vec{\sigma}$$

$$= \iint\limits_{\substack{\text{plane area}\\ \text{inside}\\ \text{the circle}}} \nabla \times \vec{V} \cdot d\vec{\sigma}$$

$$\nabla \times \vec{V} = -3\hat{k}$$

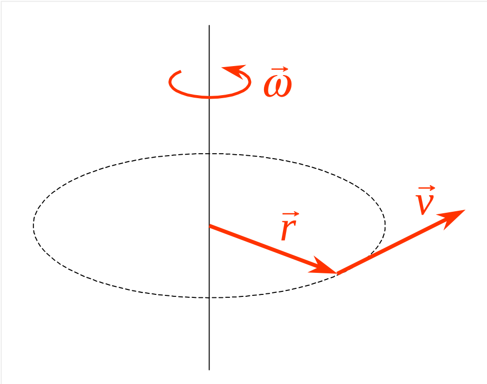
$$d\vec{\sigma} = dxdy\hat{k}$$

$$\nabla \times \vec{V} \cdot d\vec{\sigma} = -3dxdy$$

$$\nabla \times \vec{V} \cdot d\vec{\sigma} = -3 \iint\limits_{\text{circle}} dxdy = -3\pi a^2$$

Physical Meaning of Curl

Consider rotation with a constant angular velocity.



$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\nabla \times \vec{v} = \nabla \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\nabla \cdot \vec{r}) - (\vec{\omega} \cdot \nabla)\vec{r}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \implies \nabla \cdot \vec{r} = 1 + 1 + 1 = 3$$

$$(\vec{\omega} \cdot \nabla)\vec{r} = \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \vec{\omega}$$

$$\nabla \times \vec{v} = 3\vec{\omega} - \vec{\omega} = 2\vec{\omega}$$

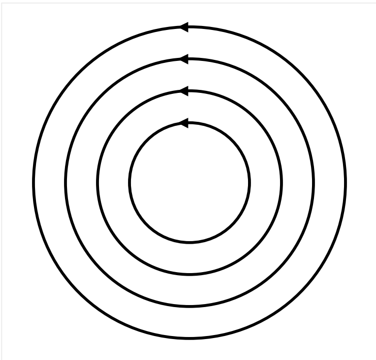
$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$$

Angular velocity!

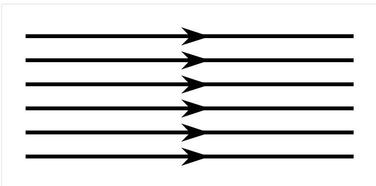
Physical Meaning of Curl

General case:

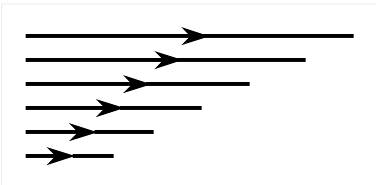
$\nabla \times \vec{V}$ at a point is a measure of the angular velocity of the fluid in the *neighbourhood* of the point.



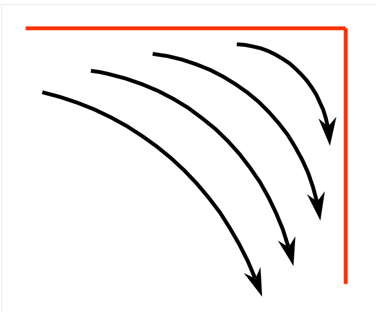
$$\nabla \times \vec{V} \neq \vec{0}$$



$$\nabla \times \vec{V} = \vec{0}$$



$$\nabla \times \vec{V} \neq \vec{0}$$



$\nabla \times \vec{V}$ may be $= \vec{0}$

Ampére's Law

Maxwell's equation:

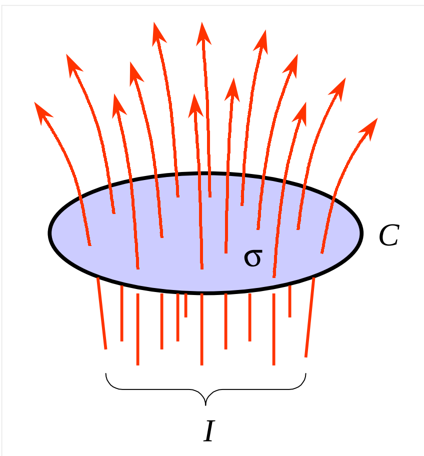
$$\nabla \times \vec{H} = \vec{J}$$

$\vec{H}$: magnetic field
$\vec{J}$: electric current density

$$\iint_\sigma (\nabla \times \vec{H}) \cdot d\vec{\sigma} = \iint_\sigma \vec{J} \cdot d\vec{\sigma} = I$$

I: total current linking C.
C: curve bounding σ.

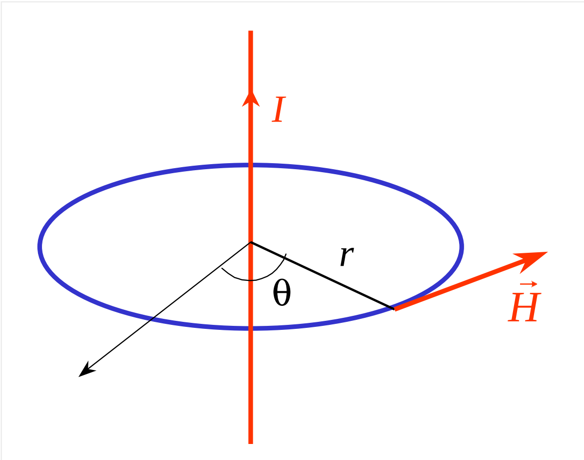
$$\iint_\sigma \nabla \times \vec{H} \cdot d\vec{\sigma} = \oint_C \vec{H} \cdot d\vec{r}$$



$$\oint_C \vec{H} \cdot d\vec{r} = I$$

Example

Magnetic field due to a straight conducting wire.



By $\nabla \times \vec{H} = \vec{J}$ and symmetry, $\vec{H}$ is along the direction shown above and $|\vec{H}|$ is the same along the circular curve in a plane $\perp$ to the wire.

$$\oint_C \vec{H} \cdot d\vec{r} = \int H(rd\theta) = 2\pi H r = I$$

$$H = \frac{I}{2\pi r}$$