# Lecture 26

## Topic:

Linear operator and matrix representation

## Relevance:

In quantum mechanics and quantum statistics (quantum statistical mechanics) as well as in certain other areas of physics, physical quantities are represented by linear operators on a vector (Hilbert) space. Mathematical operations involving linear operators are often carried out by use of matrices.

## Objective

- Understand the concept of linear operators
- Understand the properties of linear operators
- Understand the relationship between linear operator and matrix

# Linear Operators

A linearly independent set of vectors $\{\psi_i\}$ (or $\mathbf{e}_i$) spans a vector space $V_n$.

A **linear operator** on a vector space $V_n$ is a procedure for obtaining a unique vector, $\phi$, in $V_n$ for each $\psi$ in $V_n$.

$$\phi = A\psi$$

where $A$ is a linear operator.

Using the Dirac notation, we write

$$|\phi\rangle = A|\psi\rangle.$$

# Properties of Linear Operator

For linear operators $A$ and $B$, it is required that

$$A(|\psi\rangle + |\phi\rangle) = A|\psi\rangle + A|\phi\rangle$$

$$(A + B)|\psi\rangle = A|\psi\rangle + B|\psi\rangle$$

$$(AB)|\psi\rangle = A(B|\psi\rangle)$$

$$A\alpha|\psi\rangle = \alpha A|\psi\rangle$$

where $\alpha$ is a scalar. The vectors $|\psi\rangle$ and $|\phi\rangle$ are two arbitrary vectors in the vector space.

# Commutator

Linear operators, in contrast to ordinary numbers and functions, do not always commute, that is, $AB$ is not always equal to $BA$. The difference $AB - BA$ which is symbolically written as $[A, B]$ is called the **commutator** of $A$ and $B$.

In quantum mechanics, linear operators play a central role. Consider, for example, $[x, p_x]$ in the $x$-representation. Here $x$ and $p_x$, $p_x = -i\hbar\partial/\partial x$ for $i = \sqrt{-1}$ and $\hbar$ is the plank constant, represent the position and $x$-component of the momentum of a particle, respectively. The value of the commutator $[x, p_x]$ is obtained by operating on some function $\psi(x)$.

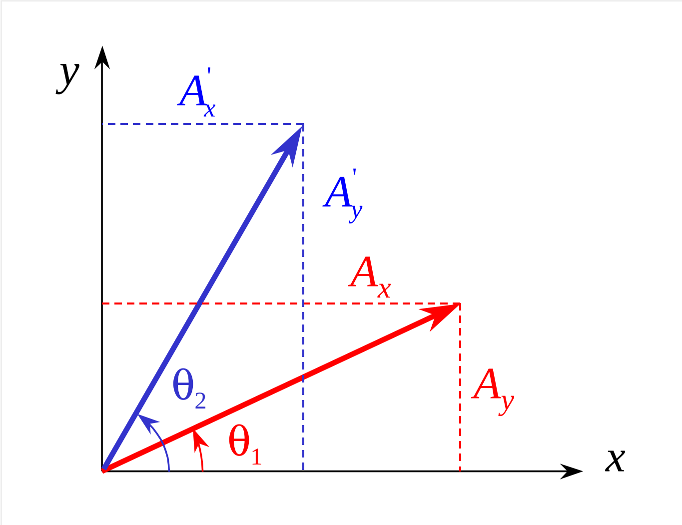
$$[x, p_x]\psi(x) = (xp_x - p_x x)\psi(x) = -i\hbar \left\{ x\frac{\partial \psi}{\partial x} - \frac{\partial (x\psi)}{\partial x} \right\}$$

$$= i\hbar\psi(x)$$

or

$$[x, p_x] = i\hbar$$

# Example




Operator: Rotate the vector $\vec{A}$ by $\theta$, from $\theta_1$ to $\theta_2$.

$$\begin{cases} A_x = A\cos\theta_1 & A'_x = A\cos\theta_2 \\ A_y = A\sin\theta_1 & A'_y = A\sin\theta_2 \end{cases}$$

Since $\theta_2 = \theta_1 + \theta$

$$\begin{aligned} A'_x = A\cos(\theta_1 + \theta) &= A(\cos\theta_1\cos\theta - \sin\theta_1\sin\theta) \\ A'_x &= A_x\cos\theta - A_y\sin\theta \\ A'_y = A\sin(\theta_1 + \theta) &= A(\sin\theta_1\cos\theta + \cos\theta_1\sin\theta) \\ A'_y &= A_y\cos\theta + A_x\sin\theta \end{aligned}$$

# Example (cont.)

$$A'_x = A_x \cos\theta - A_y \sin\theta$$
$$A'_y = A_x \sin\theta + A_y \cos\theta$$

In matrix form, this can be written as

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Or

$$\mathbf{A}' = \mathbf{R}(\theta)\mathbf{A}$$

where

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathsf{A}' = \begin{pmatrix} A'_x \\ A'_y \end{pmatrix}$$

$$\mathsf{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Operators and vectors can be represented by matrix.

# Matrix

A **matrix** is a rectangular array of quantities,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where the $a_{ij}$ are called **elements** (of the ith row and jth column); they may be real (or complex) numbers or functions.

The matrix $A$ has $m$ rows and $n$ columns and is called a matrix of **order** $m \times n$ ($m$ by $n$).

If $m = n$, the matrix is called a **square matrix**.

# Row & Column Vectors

The row matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

is called a **row vector**.

The column matrix,

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called a **column vector**.

# Matrix Representation

The operation

$$\mathbf{x}' = A\mathbf{x}$$

can be written as

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or

$$x_i' = \sum_{j=1}^{n} A_{ij} x_j$$