Lecture 26

Topic:

Linear operator and matrix representation

Relevance:

In quantum mechanics and quantum statistics (quantum statistical mechanics) as well as in certain other areas of physics, physical quantities are represented by linear operators on a vector (Hilbert) space. Mathematical operations involving linear operators are often carried out by use of matrices.

Objective

- Understand the concept of linear operators
- Understand the properties of linear operators
- Understand the relationship between linear operator and matrix

Linear Operators

A linearly independent set of vectors $\{\psi_i\}$ (or $\mathbf{e}_i$) spans a vector space V_n.

A **linear operator** on a vector space V_n is a procedure for obtaining a unique vector, ϕ, in V_n for each ψ in V_n.

$$\phi = A\psi$$

where A is a linear operator.

Using the Dirac notation, we write

$$|\phi\rangle = A|\psi\rangle.$$

Properties of Linear Operator

For linear operators A and B, it is required that

$$A(|\psi\rangle + |\phi\rangle) = A|\psi\rangle + A|\phi\rangle$$

$$(A + B)|\psi\rangle = A|\psi\rangle + B|\psi\rangle$$

$$(AB)|\psi\rangle = A(B|\psi\rangle)$$

$$A\alpha|\psi\rangle = \alpha A|\psi\rangle$$

where α is a scalar. The vectors $|\psi\rangle$ and $|\phi\rangle$ are two arbitrary vectors in the vector space.

Commutator

Linear operators, in contrast to ordinary numbers and functions, do not always commute, that is, AB is not always equal to BA. The difference $AB - BA$ which is symbolically written as $[A, B]$ is called the **commutator** of A and B.

In quantum mechanics, linear operators play a central role. Consider, for example, $[x, p_x]$ in the x-representation. Here x and p_x, $p_x = -i\hbar\partial/\partial x$ for $i = \sqrt{-1}$ and $\hbar$ is the plank constant, represent the position and x-component of the momentum of a particle, respectively. The value of the commutator $[x, p_x]$ is obtained by operating on some function $\psi(x)$.

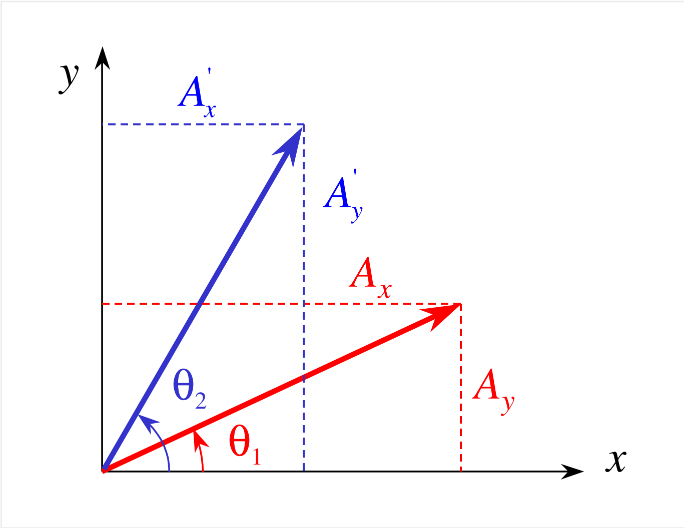
$$[x, p_x]\psi(x) = (xp_x - p_x x)\psi(x) = -i\hbar\left\{x\frac{\partial\psi}{\partial x} - \frac{\partial(x\psi)}{\partial x}\right\}$$

$$= i\hbar\psi(x)$$

or

$$[x, p_x] = i\hbar$$

Example



Operator: Rotate the vector $\vec{A}$ by θ, from θ_1 to θ_2.

$$\begin{cases} A_x = A\cos\theta_1 & A'_x = A\cos\theta_2 \\ A_y = A\sin\theta_1 & A'_y = A\sin\theta_2 \end{cases}$$

Since $\theta_2 = \theta_1 + \theta$

$$\begin{aligned} A'_x = A\cos(\theta_1+\theta) &= A(\cos\theta_1\cos\theta - \sin\theta_1\sin\theta) \\ A'_x &= A_x\cos\theta - A_y\sin\theta \\ A'_y = A\sin(\theta_1+\theta) &= A(\sin\theta_1\cos\theta + \cos\theta_1\sin\theta) \\ A'_y &= A_y\cos\theta + A_x\sin\theta \end{aligned}$$

Example (cont.)

$$A'_x = A_x \cos\theta - A_y \sin\theta$$
$$A'_y = A_x \sin\theta + A_y \cos\theta$$

In matrix form, this can be written as

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Or

$$\mathbf{A}' = \mathbf{R}(\theta)\mathbf{A}$$

where

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathsf{A}' = \begin{pmatrix} A'_x \\ A'_y \end{pmatrix}$$

$$\mathsf{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Operators and vectors can be represented by matrix.

Matrix

A **matrix** is a rectangular array of quantities,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where the a_{ij} are called **elements** (of the ith row and jth column); they may be real (or complex) numbers or functions.

The matrix A has m rows and n columns and is called a matrix of **order** $m \times n$ (m by n).

If $m = n$, the matrix is called a **square matrix**.

Row & Column Vectors

The row matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

is called a **row vector**.

The column matrix,

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called a **column vector**.

Matrix Representation

The operation

$$\mathbf{x}' = A\mathbf{x}$$

can be written as

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or

$$x_i' = \sum_{j=1}^{n} A_{ij} x_j$$